

Sec 1.6 Equilibria and the Phase Line

A differential equation of the form

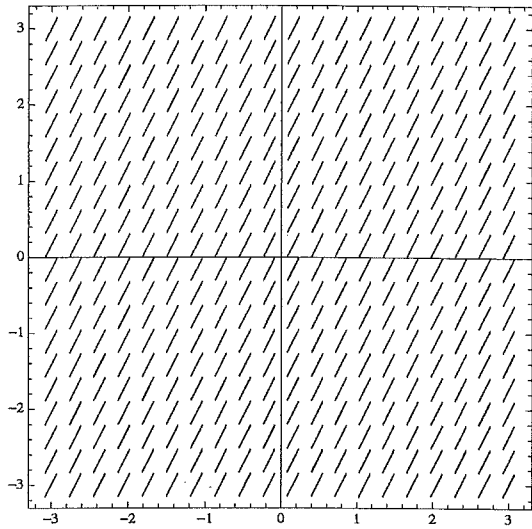
$$\frac{dy}{dt} = f(y) \quad \text{independent of } t$$

is called autonomous. The slope fields of autonomous differential equations can be represented by 1-dimensional diagrams called phase lines.

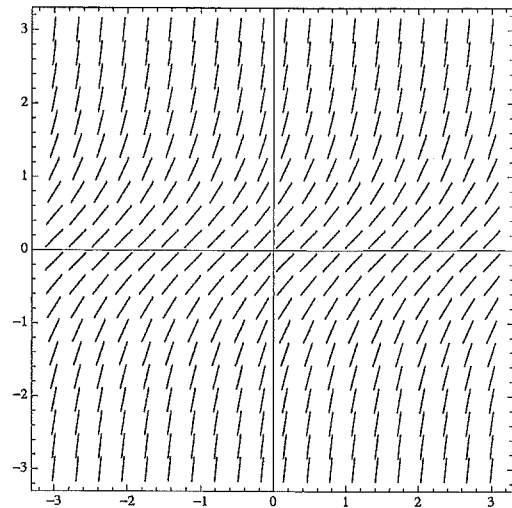
- Goals:
- ① Given an autonomous differential equation construct the phase line
 - ② Given a phase line sketch the possible solutions
 - ③ Understand the role of equilibrium points in the interpretation of phase lines
 - ④ Classify equilibrium points as sinks, sources or nodes.

Sec 1.6 Phase Lines

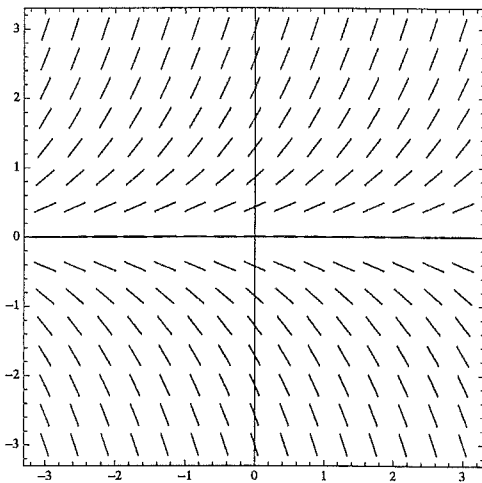
Example 1: $\frac{dy}{dt} = 2$



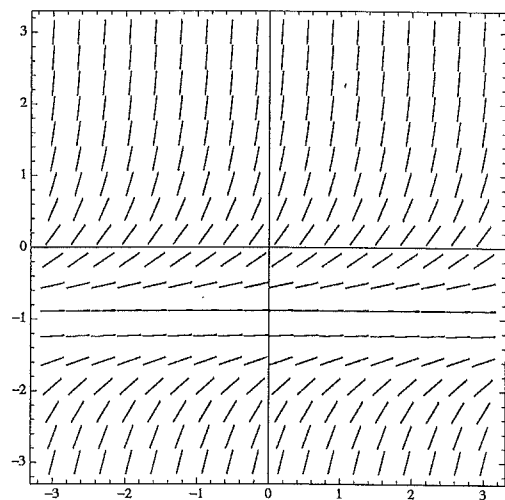
Example 1b: $\frac{dy}{dt} = 1 + y^2$



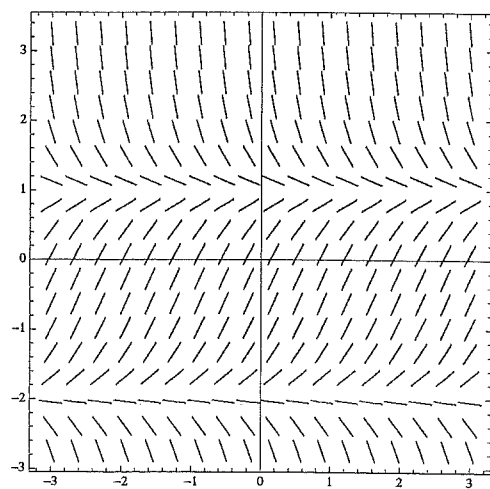
Example 2: $\frac{dy}{dt} = y$



Example 3: $\frac{dy}{dt} = (1+y)^2$



Example 4 : $\frac{dy}{dt} = (1-y)(y+2)$

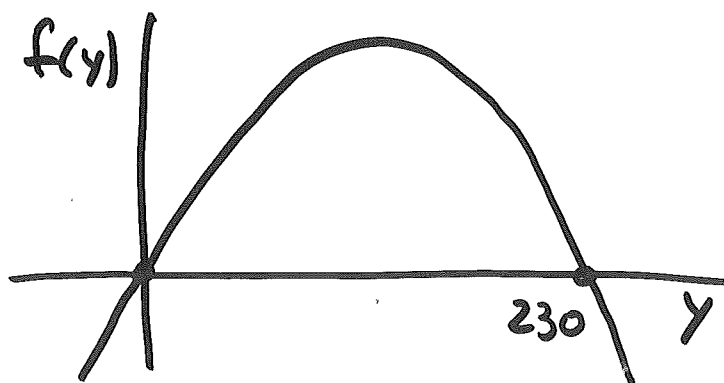


Drawing a phase line given a differential equation $\frac{dy}{dt} = f(y)$

Example $\frac{dy}{dt} = 0.4y \left(1 - \frac{y}{230}\right)$ (HW #3 Sec 1.1)

1.) Equilibrium values $y=0, y=230$

2.)



$$\frac{dy}{dt} > 0$$

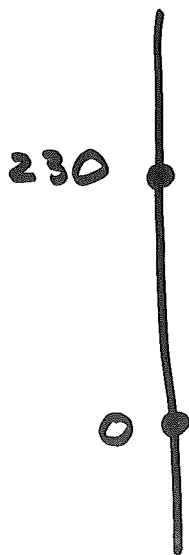
when $0 < y < 230$

$$\frac{dy}{dt} < 0$$

when $y < 0$

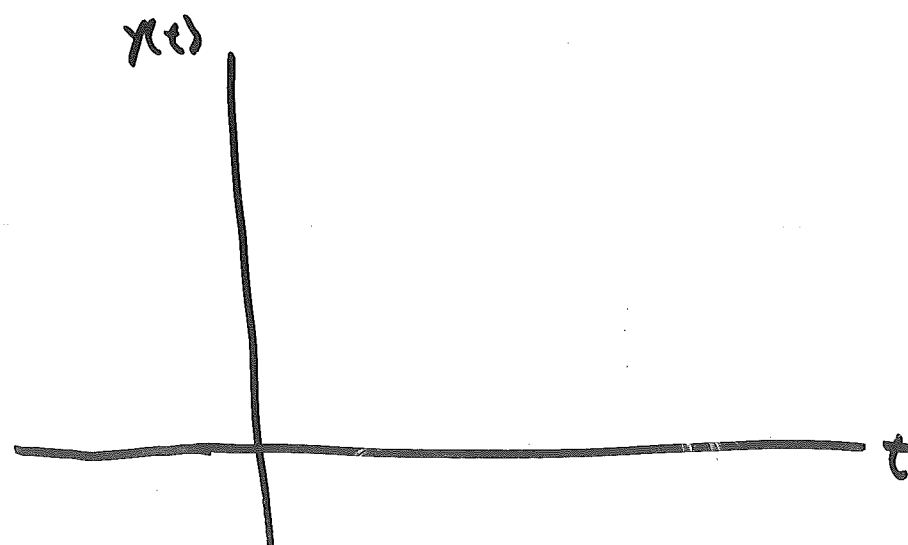
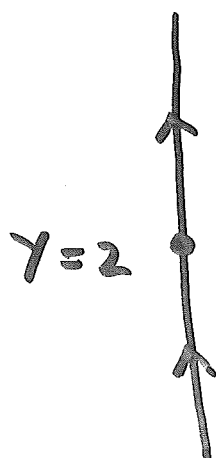
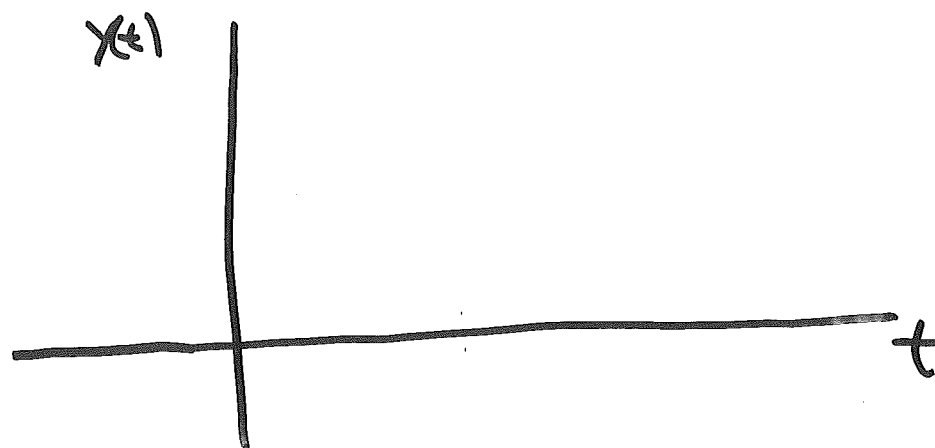
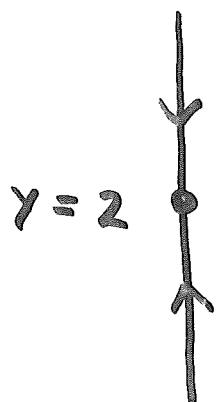
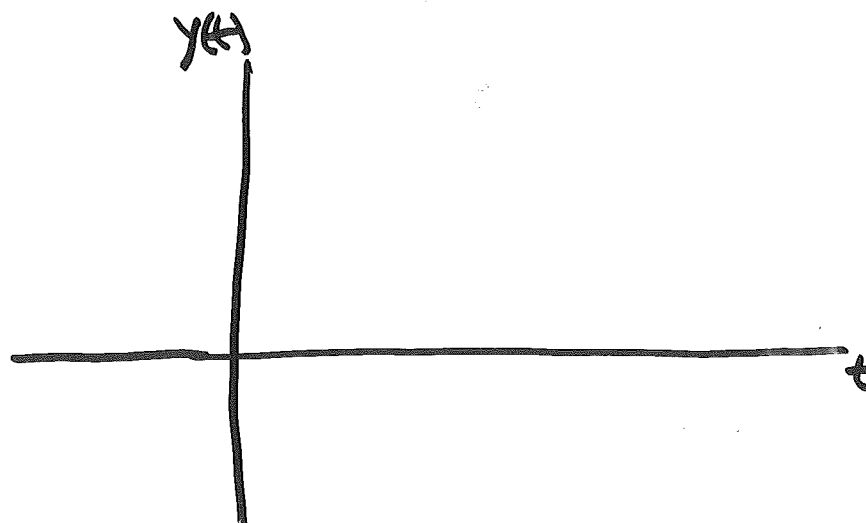
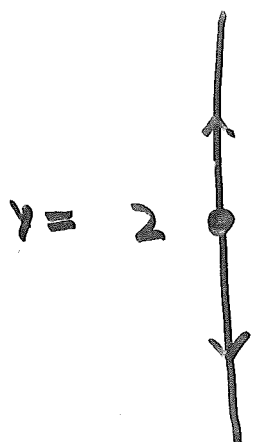
$y > 230$

3.) Phase line



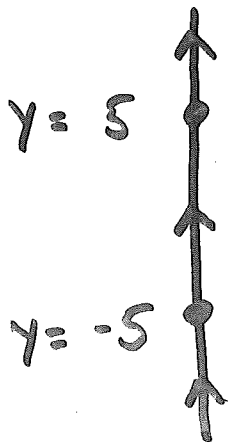
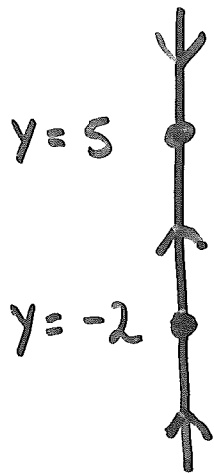
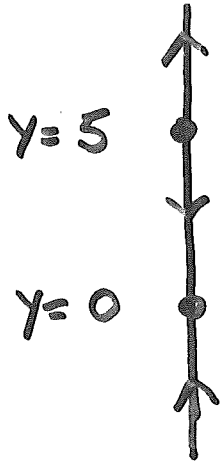
Sketching solutions given a phase line

Examples



Sketching Solutions given a phase line

more examples



General guidelines when sketching solutions from phase lines (P. 78, 79)

Suppose $\frac{dy}{dt} = f(y)$ and $y(t)$ is a solution

where $f(y)$ is continuously differentiable for all y .

- If $f(y(0)) = 0$, then $y(0)$ is an equilibrium point and $y(t) = 0$ for all t
- If $f(y(0)) > 0$, then $y(t)$ is increasing for all t and either $y(t) \rightarrow \infty$ as t increases or $y(t)$ tends to the first equilibrium point larger than $y(0)$
- If $f(y(0)) < 0$, then $y(t)$ is decreasing for all t and either $y(t) \rightarrow -\infty$ as t increases or $y(t)$ tends to the first equilibrium point smaller than $y(0)$.

Caution #1 Not all solutions exist for all time

Example: $\frac{dy}{dt} = (1+y)^2$

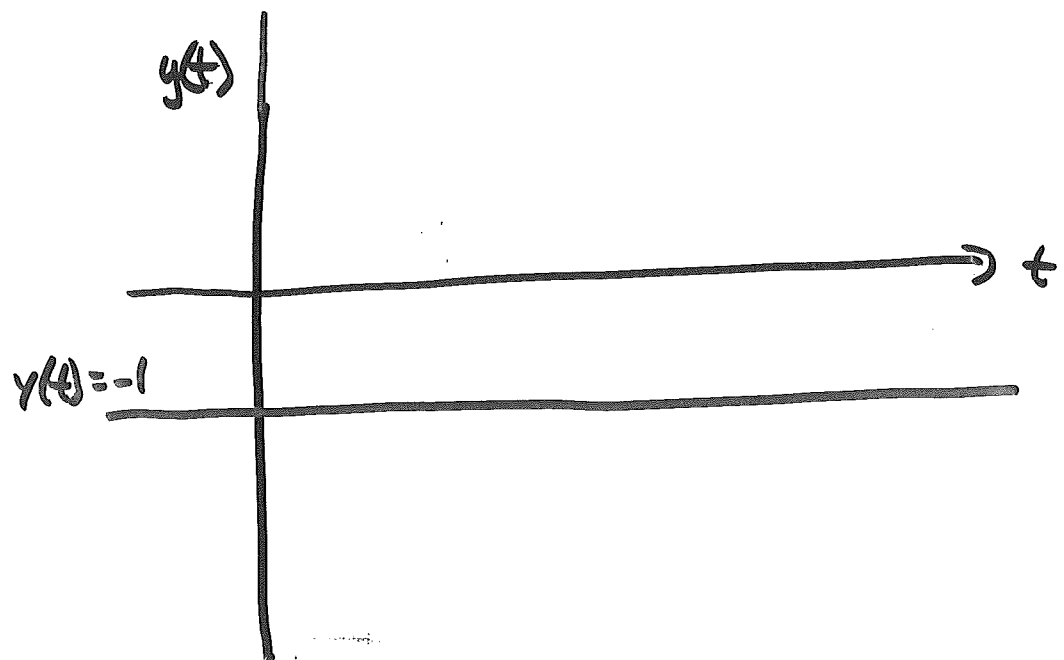
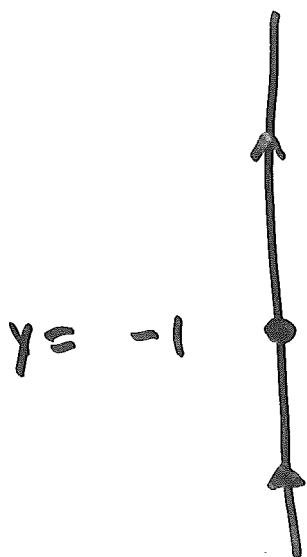
Separation of variables $\Rightarrow y(t) = -1 - \frac{1}{t+c}$

Consider initial condition $y(0)=0$

then $y(0) = 0 = -1 - \frac{1}{0+c}$ $c = -1$

$$y(t) = -1 - \frac{1}{t-1}$$

vertical asymptote at $t=1$



Caution #2 The phase line has a hole

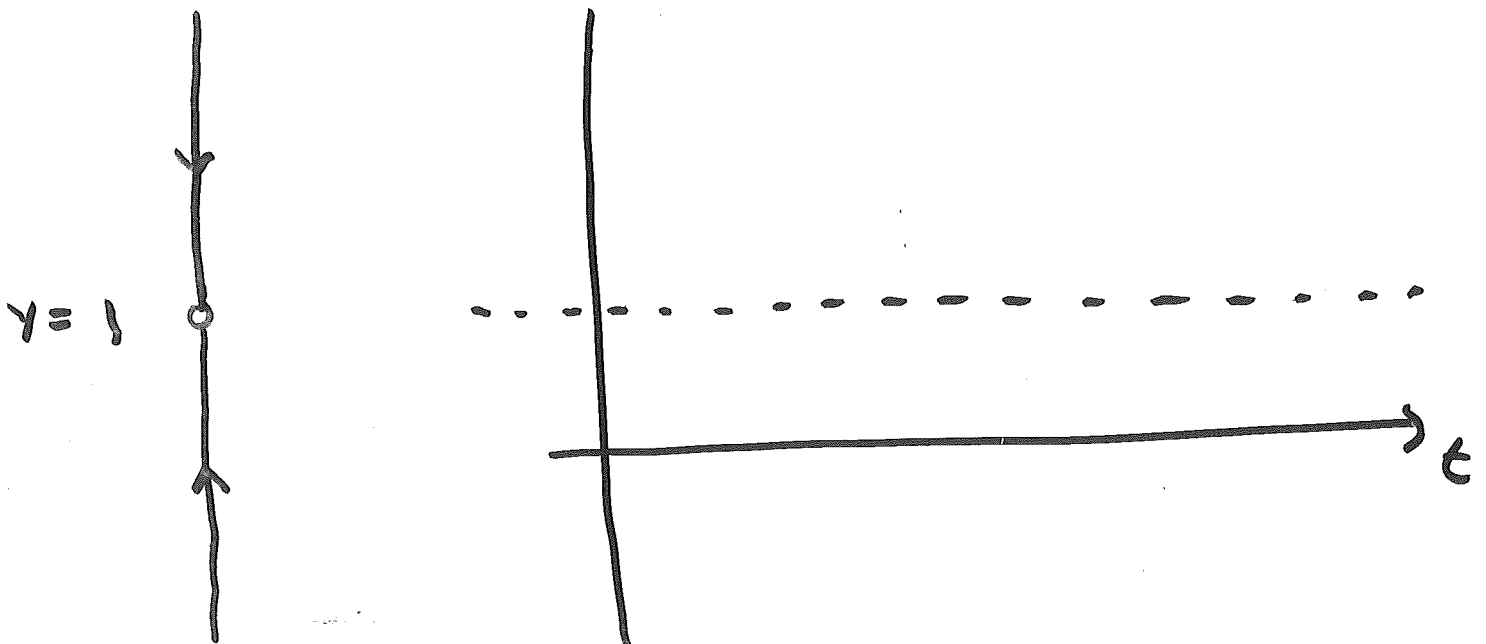
Example:

$$\frac{dy}{dt} = \frac{1}{1-y}$$

When $y = 1$, $\frac{dy}{dt}$ is undefined

$$y > 1 \quad \frac{dy}{dt} < 0$$

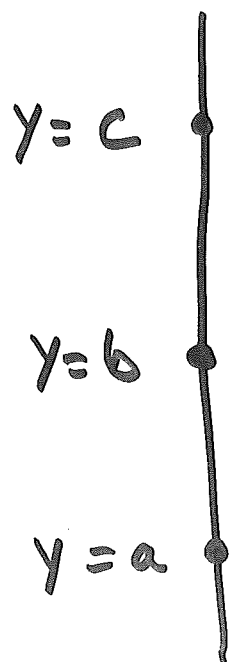
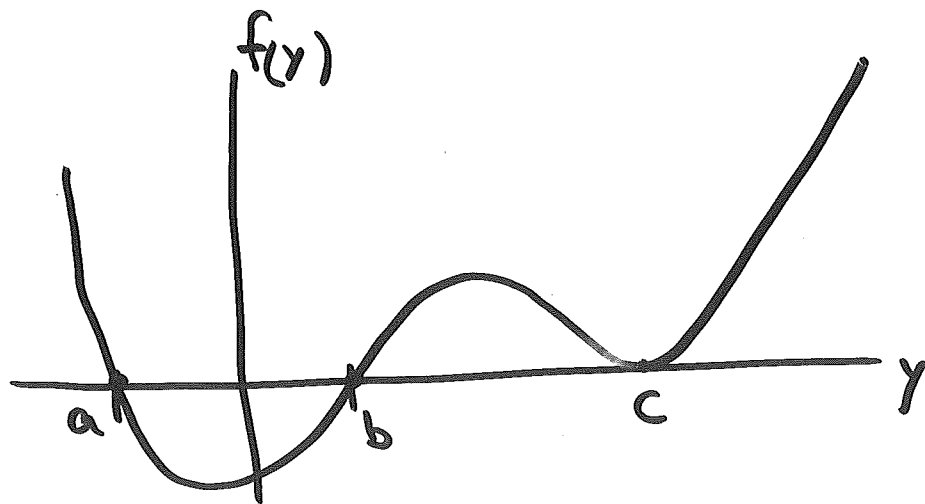
$$y < 0 \quad \frac{dy}{dt} > 0$$



Linearization Theorem

Let $\frac{dy}{dt} = f(y)$

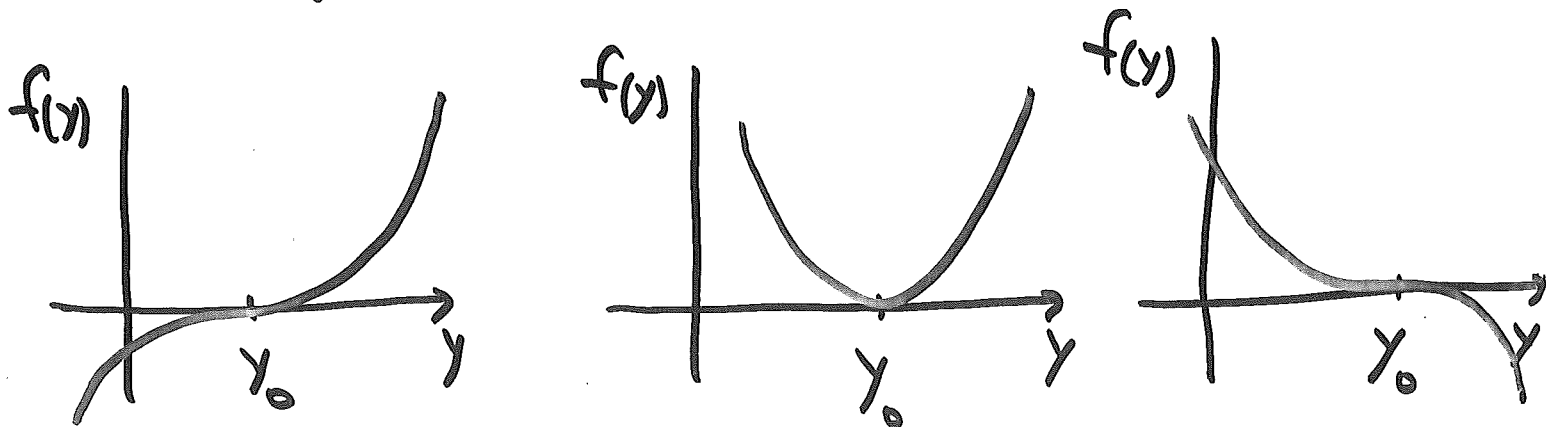
Suppose the graph of f looks like



Three cases where $f'(y) = 0$

If $\frac{dy}{dt} = f(y)$ and $f'(y_0) = 0$

then we could have one of the following:



Linearization Theorem Suppose y_0 is an equilibrium point of $\frac{dy}{dt} = f(y)$ where f is a continuously differentiable function. Then,

- If $f'(y_0) < 0$, then y_0 is a sink
- If $f'(y_0) > 0$, then y_0 is a source
- If $f'(y_0) = 0$, then we need additional information to determine the type of y_0 .

Example using Linearization

Given $\frac{dy}{dt} = h(y) = y(\cos(y^5 + 2y) - 27\pi y^4)$

We know that $y=0$ is an equilibrium point. Is it a source or a sink?