Sec 1.5

Existence and Uniqueness Theorems

An example from algebra:

Consider the polynomial function

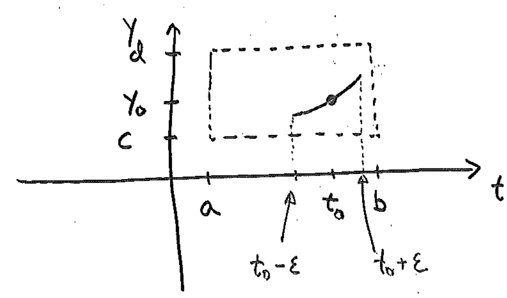
$f(x) = 2x^5 - 10x + 5$

Can we argue the existence of a zero (or root) for $f(x)$ (i.e. does it ever cross the x-axis?)

Existence Theorem: Suppose $f(t,y)$ is a continuous function in a rectangle of the form $\{(t,y) \mid a<t<b,\ c<y<d\}$ in the ty-plane.

If (t_0, y_0) is a point in this rectangle, then there exists an $\varepsilon > 0$ and a function $y(t)$ defined for $t_0-\varepsilon < t < t_0+\varepsilon$ that solves the initial value problem

$$\frac{dy}{dt} = f(t,y) \qquad y(t_0) = y_0$$

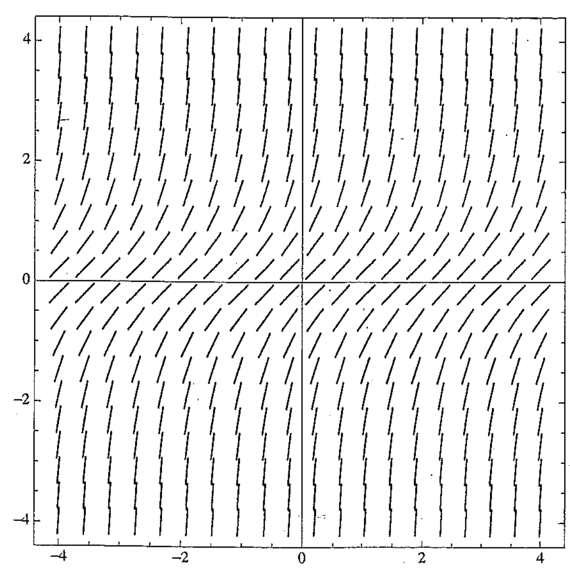


Example Consider the initial value problem

$$\frac{dy}{dt} = 1+y^2 \qquad y(0) = 0$$

Existence and Uniqueness – Example 1

$$\frac{dy}{dt} = 1+y^2$$



- In short, the Existence Theorem says that if $f(t,y)$ is continuous then there will exist a solution. However, the domain of the solution may be very small.

- The Existence Theorem does not say that if $f(t,y)$ is not continuous there cannot be a solution, it just does not guarantee that a solution exists.

Uniqueness Theorem

Suppose $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in a rectangle of the form $\{(t,y) \mid a < t < b,\ c < y < d\}$ in the ty-plane. If (t_0, y_0) is a point in the rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial value problem

$$\frac{dy}{dt} = f(t,y) \qquad y(t_0) = y_0$$

for all t in the interval $t_0 - \varepsilon < t < t_0 + \varepsilon$ (where ε is some positive number), then

$$y_1(t) = y_2(t)$$

for all t in the interval $t_0 - \varepsilon < t < t_0 + \varepsilon$. That is, the solution to the initial value problem is unique.

Lack of uniquness - an example

$$\frac{dy}{dt} = 3y^{2/3}$$

$f(t,y)$ is continuous in (t,y) plane

$$\frac{\partial f}{\partial y} = 2y^{-\frac{1}{3}} = \frac{2}{y^{\frac{1}{3}}}$$

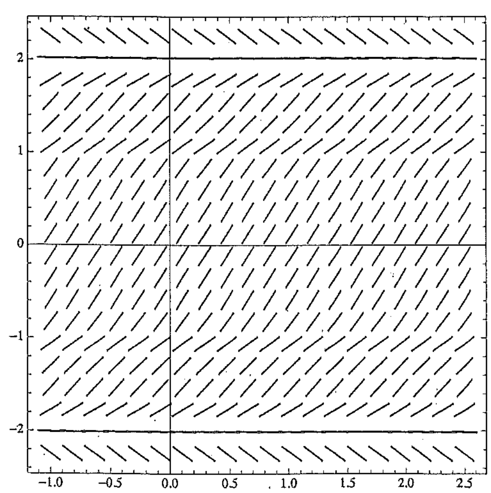
is not continuous at $y = 0$

Consider the Initial Value Problem

$$\frac{dy}{dt} = 3y^{2/3} \qquad y(0) = 0$$

Existence and Uniqueness – Example 2

$$\frac{dy}{dt} = \frac{(y^2-4)(\sin^2 y^3 + \cos y - 2)}{2}$$



Existence and Uniqueness –Example 3

$$\frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2} \qquad y(0) = 0$$