

Sec 1.2 - Separation of Variables

Defn: A first order ordinary differential equation has the form

$$\frac{dy}{dt} = f(t, y)$$

Examples

$$\frac{dy}{dt} = y$$

$$\frac{dy}{dt} = t + y$$

$$\frac{dy}{dt} = \frac{t+1}{y^2}$$

Defn: A solution of a differential eqn is a function $y(t)$ that when substituted into the diff. eqn. satisfies the equation for all values of t .

Example: $\frac{dy}{dt} = 3y$

Solutions include: $y(t) = e^{3t}$

$$y(t) = 0$$

$$y(t) = 2e^{3t}$$

general solution $\rightarrow y(t) = Ce^{3t}$ for some real number C

Example: $\frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$

solutions: $y(t) = 1+t$

Equilibrium Solutions:

Ask for values of y that make $f(y, t) = 0$

$$f(y, t) = \frac{y^2-1}{t^2+2t}$$

$$\Rightarrow y(t) = 1 \quad \text{or} \quad y(t) = -1$$

Defn: A differential equation together with an initial condition is called an Initial Value Problem (I.V.P)

Example: $\frac{dy}{dt} = 12t^3 - 2\sin t$ $y(0) = 3$

$$y(t) = 3t^4 + 2\cos t + C$$

$$y(0) = 3 \cdot 0^4 + 2\cos(0) + C = 3$$

Defn: A differential equation $\frac{dy}{dt} = f(t, y)$ is called separable if the function

$f(t, y)$ can be written as a product of two functions: one that depends on t and one that depends on y , $f(t, y) = g(t) \cdot h(y)$

Example 1: $\frac{dy}{dt} = 3y$

$$f(t, y) = 3 \cdot y \quad \text{let } g(t) = 3 \quad h(y) = y$$

Example 2

$$\frac{dy}{dt} = \frac{t}{y^2}$$

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Formal Approach

$$\frac{dy}{dt} = h(y) \cdot g(t)$$

$$\frac{1}{h(y)} \frac{dy}{dt} = g(t)$$

$$\int \frac{1}{h(y)} \cdot \frac{dy}{dt} \cdot dt = \int g(t) dt$$

Method of substitution:

$$\text{Let } u = y(t) \quad du = \frac{dy}{dt} \cdot dt$$

$$\int \frac{1}{h(y)} du = \int g(t) dt$$

dummy
variable

$$\int \frac{1}{h(y)} dy = \int g(t) dt$$

Informal Approach

$$\frac{dy}{dt} = h(y) \cdot g(t)$$

$$\frac{1}{h(y)} \cdot dy = g(t) dt$$

$$\int \frac{1}{h(y)} dy = \int g(t) dt$$

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Separation of Variables - Caution Missing Solutions

Example:

$$\frac{dy}{dt} = y^2$$

$$\frac{1}{y^2} dy = dt$$

$$-\frac{1}{y} = t + C$$

$$-y = \frac{1}{t+C}$$

$$y(t) = \frac{-1}{t+C}$$

General Solution

$$y(t) = \left\{ \begin{array}{l} \frac{-1}{t+C} \\ 0 \end{array} \right.$$

consider initial condition

$$y(0) = 4$$

consider initial condition
 $y(0) = 0$

Separation of Variables - Getting Stuck

Ex 1

$$\frac{dy}{dt} = \frac{y}{1+y^2}$$

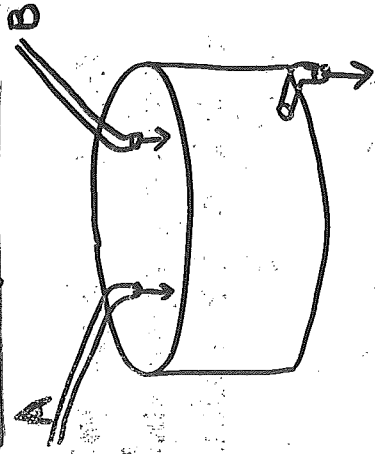
$$\frac{1+y^2}{y} dy = dt$$

Ex 2

$$\frac{dy}{dt} = \sec(y^2)$$

A mixing problem

(P.32)



Consider a large vat containing sugar water that is used to make soft drinks.

- The vat contains 100 gal of liquid. The amount in = amount out
- The vat is kept well mixed so the sugar concentration is uniform throughout the vat.
- ✓ Sugar water containing 5 Tsp of sugar per gal enters the vat through pipe A at a rate of 2 gal/min
- Sugar water containing 10 Tsp of sugar per gallon enters the vat through pipe B at a rate of 1 gal/min
- Sugar leaves the vat through pipe C at a rate of 3 gal/min.

Find, the amount of sugar in the vat at time t .