Sec 1.2 - Separation of Variables

Defn: A first order ordinary differential equation has the form

$$\frac{dy}{dt} = f(t,y)$$

Examples

$$\frac{dy}{dt} = y$$

$$\frac{dy}{dt} = t + y$$

$$\frac{dy}{dt} = \frac{t+1}{y^2}$$

Defn: A solution of a differential eqn is a function $y(t)$ that when substituted into the diff. eqn. satisfies the equation for all values of t.

<u>Example:</u> $\frac{dy}{dt} = 3y$

Solutions include: $y(t) = e^{3t}$

$y(t) = 0$

$y(t) = 2e^{3t}$

general solution $\rightarrow$ $y(t) = c\,e^{3t}$ for some real number c

<u>Example:</u> $\frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$

solutions : $y(t) = 1+t$

<u>Equilibrium Solutions:</u>

Ask for values of y that make $f(y,t) = 0$

$f(y,t) = \frac{y^2-1}{t^2+2t}$ $\Rightarrow$ $y(t) = 1$ or $y(t) = -1$

<u>Defn</u>: A differential equation together with an initial condition is called an <u>Initial Value Problem</u> (I.V.P)

<u>Example</u>: $\frac{dy}{dt} = 12t^3 - 2\sin t$ $\quad y(0) = 3$

$$y(t) = 3t^4 + 2\cos t + C$$

$$y(0) = 3\cdot 0^4 + 2\cos(0) + C = 3$$

<u>Defn</u>: A differential equation $\frac{dy}{dt} = f(t,y)$ is called <u>seperable</u> if the function $f(t,y)$ can be written as a product of two functions: one that depends on t and one that depends on y, $f(t,y) = g(t)\cdot h(y)$

<u>Example 1</u>: $\frac{dy}{dt} = 3y$

$f(t,y) = 3 \cdot y$ Let $g(t) = 3$ $h(y) = y$

Example 2

$$\frac{dy}{dt} = \frac{t}{y^2}$$

Formal Approach

$$\frac{dy}{dt} = h(y) \cdot g(t)$$

$$\frac{1}{h(y)} \frac{dy}{dt} = g(t)$$

$$\int \frac{1}{h(y(t))} \cdot \frac{dy}{dt} \cdot dt = \int g(t)\,dt$$

Method of substitution:

Let $u = y(t)$ $\quad du = \frac{dy}{dt} \cdot dt$

$$\int \frac{1}{h(u)}\,du = \int g(t)\,dt$$

dummy variable

$$\int \frac{1}{h(y)}\,dy = \int g(t)\,dt$$

Informal Approach

$$\frac{dy}{dt} = h(y) \cdot g(t)$$

$$\frac{1}{h(y)} \cdot dy = g(t)\,dt$$

$$\int \frac{1}{h(y)}\,dy = \int g(t)\,dt$$

Separation of Variables - Caution Missing Solutions

Example:

$$\frac{dy}{dt} = y^2$$

$$\frac{1}{y^2}dy = dt$$

$$-\frac{1}{y} = t + c$$

$$-y = \frac{1}{t+c}$$

$$y(t) = \frac{-1}{t+c}$$

General Solution

$$y(t) = \begin{cases} \dfrac{-1}{t+c} \\ 0 \end{cases}$$

consider initial condition
$y(0) = 4$

consider initial condition $y(0) = 0$

Separation of Variables - Getting Stuck

Ex 1

$$\frac{dy}{dt} = \frac{y}{1+y^2}$$

$$\frac{1+y^2}{y}\,dy = dt$$

Ex 2

$$\frac{dy}{dt} = \sec(y^2)$$

A mixing problem

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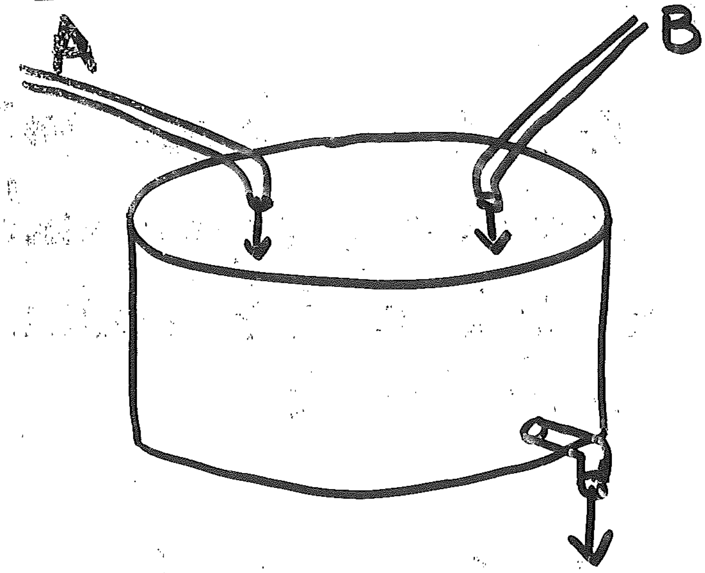

Consider a large vat containing sugar water that is used to make soft drinks.

- The vat contains 100 gal of liquid. The amount in = amount out
- The vat is kept well mixed so the sugar concentration is uniform throughout the vat.
- Sugar water containing 5 Tsp of sugar per gal enters the vat throug pipe A at a rate of 2 gal/min
- Sugar water containing 10 Tsp of sugar per gallon enters the vat through pipe B at a rate of 1 gal/min
- Sugar leaves the vat through pipe C at a rate of 3 gal/min.

Find, the amount of sugar in the vat at time t.