<u>Problem</u>: The population of rabbits on a deserted island doubles every 4 months. Initially there are only 6 rabbits on the island. What will the population of the rabbits be in 5 years?

Assumption from population dynamics:

$\boxed{\text{The growth rate of a population is proportional to the size of the population.}}$

Let $y(t)$ = population at time t

$\frac{dy}{dt}$ = rate of growth of the populati[on]

$$\frac{dy}{dt} \propto y$$

symbol for proportionality

An equation that relates a function and its derivative(s) is called a <u>differential equation</u>.

<u>Examples</u>:

$$\frac{dy}{dt} = 3y$$

$$m\frac{d^2y}{dt^2} = ky$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = \cos(3t)$$

<u>Solutions</u> of differential equations are <u>function[s]</u>

t is an independent variable
y is a function of t
m, k, b, a are parameters

Differential Equations are used to model
- population growth
- mechanical structures
- economic behavior
- electronic circuits

<u>Caution</u>: In general most differential eqns cannot be solved analytically. (We cannot find expression for the functions that solve the equation.) However, the theory of differential equations gaurentees the existenc[e] of solutions (in most cases) and computer technology alows us to view the graphs of the solutions

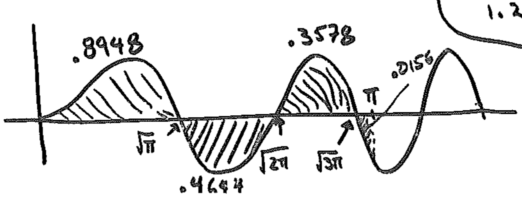
Recall that we had a similar situation when we studied integration

$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = -\cos(\pi) - -\cos(0)$$
$$= --1 + 1 = 2$$

$$\int_0^{\pi} x \cdot \sin x\,dx =$$

$$\int_0^{\pi} \sin(x^2)\,dx =$$

$$\begin{array}{r} .8948 \\ +\ .3578 \\ \hline 1.2526 \end{array} \qquad \begin{array}{r} .4644 \\ +\ .0156 \\ \hline .4790 \end{array}$$



$$\begin{array}{r} 1.2526 \\ -\ .4790 \\ \hline .7736 \end{array}$$

Example 1 Unlimited Population Growth

The assumption that the rate of growth of a population is proportional to the size of the population leads to the equation:

$$\frac{dy}{dt} = ky$$ where k is a proportionality constant

Possible Solutions

Consider the US Census figures in Table 1.1

Let $y(t)$ = population in U.S. at time t (years)

$t = 0 \Rightarrow$ year 1790

Assume unlimited population growth

$$\frac{dy}{dt} = ky \qquad y(0) = 3.9 \text{ million}$$

General solution $y(t) = Ae^{kt}$

$$y(0) = Ae^{k \cdot 0} = A$$

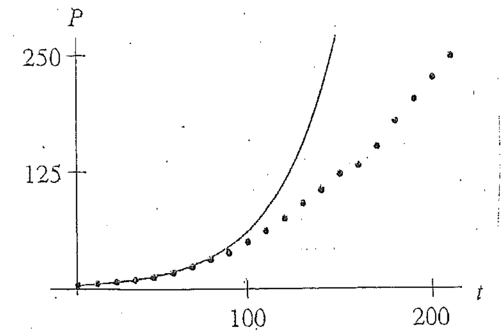
We need a second data point to find k

Let's use in year 1800 population = 5.3 mil

$$y(10) = 5.3$$

Table 1.1 US Census Figures (millions)

t	Year	Actual	$P(t) =$
0	1790	3.9	
10	1800	5.3	
20	1810	7.2	
30	1820	9.6	
40	1830	13	
50	1840	17	
60	1850	23	
70	1860	31	
80	1870	39	
90	1880	50	
100	1890	63	



Limited resources and the Logistic Model

Let N = carrying capacity of the environment for a specific species. That is the largest population that the environment can support.

Two assumptions

1.) If the population is small (relative to N) then the rate of growth of the population is proportional to its size

$$\frac{dP}{dt} \approx kP$$

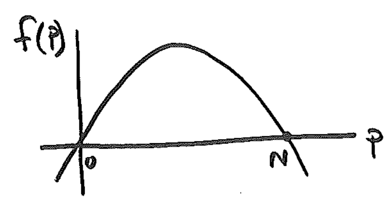
2.) If $P > N$ then $\frac{dP}{dt} < 0$ rate of growth is negative

If $P < N$ then $\frac{dP}{dt} > 0$

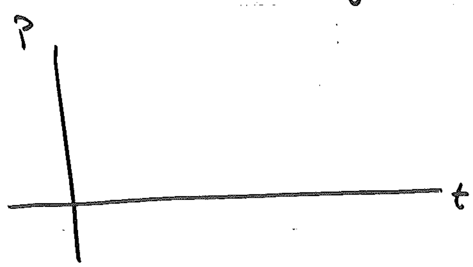
Qualitative Analysis of the logistic model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) = f(P) \qquad \begin{matrix} k > 0 \\ N > 0 \end{matrix}$$

$$f(P) = kP\left(1 - \frac{P}{N}\right) = -\frac{k}{N}P^2 + kP$$



Possible Solutions of the Logistic Model



Example 3 - System of differential equations
Predator - Prey Model

Consider an island where rabbits (R) have unlimited food but the foxes (F) can only eat rabbits.

Assumptions

1.) If $F = 0$, the rabbit reproduce at a rate proportional to its population size

2.) The rate at which the rabbits are eaten is proportional to the number of fox and rabbit interactions

3.) If $R = 0$, the fox population declines at a rate which is proportional to itself

4.) The rate at which foxes are born is proportional to the number of rabbits eaten (i.e. proportional to the number of rabbit and fox interactions.