EE 445/645: Physical Models in Remote Sensing (Spring 2026)

Chapter 02-Part 02: The Extinction Coefficient – Notes (See C2-P2 PPTs)

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§1. Introduction.

In this chapter we shall derive the extinction coefficient for a vegetation canopy idealized as a turbid medium. The extinction coefficient is the fundamental quantity that governs the attenuation of radiation as it propagates through the canopy; its determination is therefore prerequisite to the solution of any radiative transfer problem in vegetated media.

We shall find that the extinction coefficient admits a natural factorization into two quantities of distinct physical character: the leaf area density, which describes the amount of foliage per unit volume, and the *geometry function* G, which encodes the angular distribution of leaf orientations. The geometry function, first introduced by Ross (1981), is remarkable in that it renders the extinction coefficient dependent upon the direction of photon travel but independent of wavelength—a property that distinguishes radiative transfer in vegetation from that in all other natural media.

After deriving the extinction coefficient and establishing the normalization of the G-function, we shall examine its properties, discuss the limitations of the turbid-medium formulation, present explicit analytical forms for several classical leaf-angle distributions, and treat the special case of coniferous canopies in which needles are clustered into shoots. Complete derivations of all results quoted in §8 are given in Appendices A, B, and C.

§2. The turbid-medium abstraction.

Canopies of natural vegetation exhibit formidable complexity. Branches twist and turn in every direction, leaves overlap and cluster at irregular intervals, and the architecture changes from season to season. Rather than attempt to capture this complexity in all its detail, we resort to a simple abstraction: the canopy is idealized as a medium densely filled with small planar elements of negligible thickness and area. This representation is referred to as a turbid medium, or, colloquially, a green gas (Figure 1).

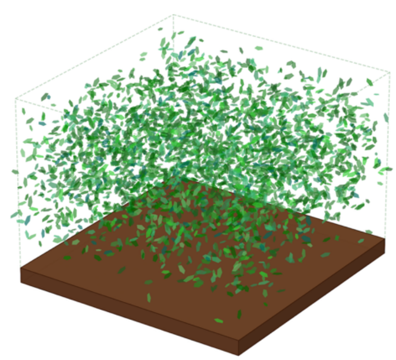

Figure 1. Vegetation canopy idealized as a turbid of green planar elements.

The abstraction rests upon the following assumptions. First, the canopy medium is filled with a sufficiently large number of small planar elements that macroscopic variables such as the leaf area density and the leaf normal orientation distribution may be treated as continuous and differentiable functions of position. Second, each element is oriented in space according to a probability density function $g_L(\mathbf{r}, \mathbf{\Omega}_L)$ defined over the hemisphere of leaf-normal directions $\mathbf{\Omega}_L$. Third, and most critically, the projections of these elements onto any plane perpendicular to an arbitrary direction $\mathbf{\Omega}$ do not overlap. It is this no-overlap assumption that permits us to compute the total shadow area as the arithmetic sum of the individual shadow areas, and it forms the foundation of the entire derivation that follows.

§3. Definition of the extinction coefficient.

The *extinction coefficient* (total interaction cross section) is defined in terms of the probability that a photon, while traveling an infinitesimal distance $d\xi$ through the medium, will interact with the elements of the host medium. This probability is given by the product

$$\sigma(\mathbf{r}, \mathbf{\Omega})\, d\xi,$$

where $\sigma(\mathbf{r}, \mathbf{\Omega})$ is the extinction coefficient at position $\mathbf{r}$ for the direction of photon travel $\mathbf{\Omega}$. It has dimensions of inverse length (m^{-1}).

The quantity σ thus measures, per unit distance traveled, the likelihood that a photon will be intercepted by the foliage. A large value of σ indicates an optically dense medium in which photons are rapidly absorbed or scattered; a small value indicates a relatively transparent medium. We note that σ depends upon both position, because the canopy may be denser in some regions than in others, and upon the direction of photon travel, because the projected area of the leaves varies with the direction from which they are viewed.

§4. Derivation of the extinction coefficient.

Consider an elementary volume $\Delta V = dS\, d\xi$ at position $\mathbf{r}$ in the canopy. This small cylinder has cross-sectional area dS perpendicular to the direction of photon travel $\mathbf{\Omega}$ and depth $d\xi$ along $\mathbf{\Omega}$. We suppose that the volume contains a sufficient number of small planar leaf elements of negligible thickness.

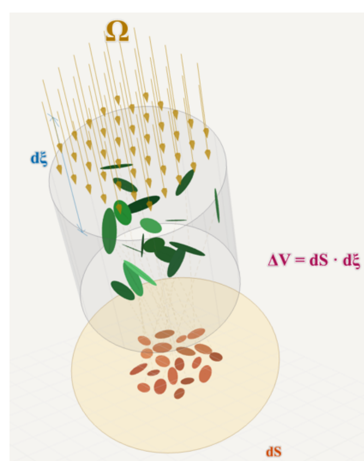

Figure 2. Projection of leaf elements in an elementary volume ΔVat $\mathbf{r}$ on to a plane perpendicular to the direction of photon travel $\mathbf{\Omega}$.

The probability that a photon comprising the incident beam will collide with a leaf element in this volume is equal to the ratio (Figure 2)

$$\sigma(\mathbf{r},\mathbf{\Omega})\,d\xi = \frac{\text{total shadow area of leaves on a plane perpendicular to } \mathbf{\Omega}}{dS}.$$

We now express the total shadow area explicitly. Let the volume contain N leaf elements, the ith of which has one-sided area s_i and leaf-normal direction $\mathbf{\Omega}_{Li}$. The shadow (projected area) of this element on a plane perpendicular to $\mathbf{\Omega}$ is $s_i\,|\mathbf{\Omega}_{Li}\cdot\mathbf{\Omega}|$. Since, by assumption, the individual shadows do not overlap, the total shadow area is the arithmetic sum

$$\sigma(\mathbf{r},\mathbf{\Omega})\,d\xi = \frac{\sum_{i=1}^{N} s_i\;|\mathbf{\Omega}_{Li}\cdot\mathbf{\Omega}|}{dS}.$$

Counting over individual elements is tedious when N is large. It is more convenient to group the elements by the direction of their normals (Figure 3).

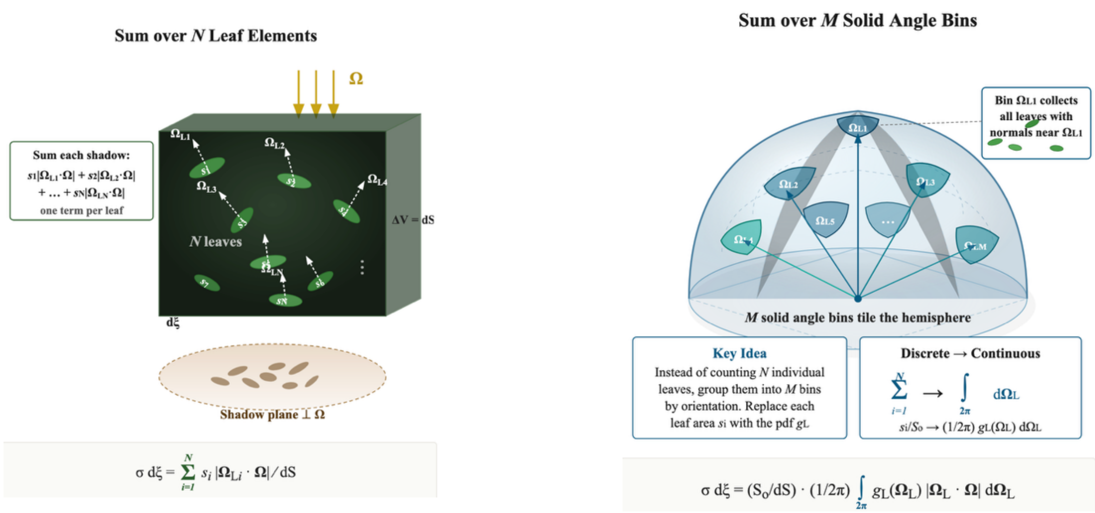

Figure 3. Two equivalent ways to compute the extinction cross-section σ of a vegetation canopy modeled as a turbid medium. Left panel shows the direct approach: each of the N leaf elements (areas s_1, s_2, …, s_N) in the elementary volume is individually projected onto a plane perpendicular to the photon travel direction $\mathbf{\Omega}$, and their shadow areas are summed term by term. The projection of leaf i with normal $\mathbf{\Omega}_{Li}$ contributes $s_i\,|\mathbf{\Omega}_{Li}\cdot\mathbf{\Omega}|$ to the total shadow. This is exact but impractical when N is large. Right panel shows the efficient alternative: instead of enumerating leaves one by one, the upper hemisphere of leaf normal directions is partitioned into M solid angle bins, each collecting all leaves whose normals fall within that bin. The fractional area of leaves in bin $\mathbf{\Omega}_{Li}$ is expressed through the leaf angle probability density $g_L(\mathbf{\Omega}_L)$, so the discrete sum over N leaves becomes a continuous integral over the hemisphere.

Let the hemisphere of leaf-normal directions be partitioned into M solid-angle increments $d\mathbf{\Omega}_{Li}$, $i = 1, \ldots, M$. The ratio of the total one-sided area of all elements whose normals fall within $d\mathbf{\Omega}_{Li}$ to the total leaf area S_o in the volume is, by definition, the probability of finding leaf elements of that orientation:

$$\frac{s_i(\mathbf{\Omega}_{Li})}{S_o} = \frac{1}{2\pi}\,g_L(\mathbf{r},\mathbf{\Omega}_{Li})\,d\mathbf{\Omega}_{Li},$$

whence

$$s_i(\mathbf{\Omega}_{Li}) = \frac{S_o}{2\pi}\, g_L(\mathbf{r}, \mathbf{\Omega}_{Li})\, d\mathbf{\Omega}_{Li}.$$

Here g_L is the probability density function of leaf-normal orientation. Substituting into the sum and passing to the limit as $M \to \infty$, we obtain

$$\sigma(\mathbf{r}, \mathbf{\Omega})\, d\xi = \frac{S_o}{dS}\left[\frac{1}{2\pi}\int_0^{2\pi} g_L\,(\mathbf{r}, \mathbf{\Omega}_L)\,|\mathbf{\Omega}_L \cdot \mathbf{\Omega}|\, d\mathbf{\Omega}_L\right].$$

We now identify the two factors on the right-hand side. The ratio $S_o/(dS\, d\xi)$ is the total one-sided leaf area per unit volume, which we denote by $u_L(\mathbf{r})$ and call the leaf area density (m^{-1}). The bracketed integral depends only upon the leaf-normal distribution and the direction of photon travel; we denote it by $G(\mathbf{r}, \mathbf{\Omega})$ and call it the geometry function, or G-function. Thus

$$\sigma(\mathbf{r}, \mathbf{\Omega}) = u_L(\mathbf{r})\, G(\mathbf{r}, \mathbf{\Omega}),$$

where

$$G(\mathbf{r}, \mathbf{\Omega}) = \frac{1}{2\pi}\int_0^{2\pi} g_L\,(\mathbf{r}, \mathbf{\Omega}_L)\,|\mathbf{\Omega}_L \cdot \mathbf{\Omega}|\, d\mathbf{\Omega}_L.$$

This factorization is of fundamental importance: it cleanly separates the quantity of foliage, u_L, from the geometric arrangement of that foliage, G.

§5. The normalization of the G-function.

The G-function, also called the geometry factor, was first introduced by Juhan Ross in his classical treatise *The Radiation Regime and Architecture of Plant Stands* (1981). It satisfies an important normalization condition which we now derive.

Averaging the G-function over all directions of photon travel $\mathbf{\Omega}$ uniformly distributed over the hemisphere of 2π steradians, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} G\,(\mathbf{r}, \mathbf{\Omega})\, d\mathbf{\Omega} = \frac{1}{2}.$$

The proof proceeds by substituting the definition of G and interchanging the order of integration. Writing $\mathbf{\Omega}_L \cdot \mathbf{\Omega} = \cos\Theta$, where Θ is the angle between the leaf normal and the direction of photon travel, the inner integral over all directions $\mathbf{\Omega}$ of the quantity $|\cos\Theta|$ over the full sphere of 4π steradians yields 2π. The outer integral over $\mathbf{\Omega}_L$ then reduces to the normalization of g_L itself. A rigorous proof is given in Appendix B, §22–§24, where we establish the master identity $\int_0^{\pi/2} \psi\,(\mu, \mu_L)\sin\theta\, d\theta = 1/2$ and show that the normalization of G reduces to the normalization of $\overline{g}_L$.

The physical content of this result is noteworthy: the average projected area, when averaged uniformly over all possible directions of illumination, is exactly one-half of the total one-sided leaf area. This holds regardless of the leaf orientation distribution—whether the leaves are predominantly horizontal, predominantly vertical, or distributed in any other fashion.

§6. Properties of the G-function and the extinction coefficient.

The G-function and the extinction coefficient σ exhibit several properties that distinguish radiative transfer in vegetation canopies from that in other natural media.

Directional dependence. Both G and σ depend explicitly upon the direction of photon travel $\mathbf{\Omega}$. The single exception occurs when the leaf-normal distribution is uniform, $g_L(\mathbf{r}, \mathbf{\Omega}_L) = 1$, in which case $G = 1/2$ independent of direction (see Appendix A, §2–§4, for the derivation).

Wavelength independence. The extinction coefficient is independent of photon frequency (wavelength), because the leaves and other organs of a vegetation canopy are much larger than the wavelengths of solar radiation. Extinction is a purely geometric phenomenon: it depends upon shapes and orientations, not upon wavelength. This property is the converse of that encountered in atmospheric and oceanic media, where the extinction coefficient is typically wavelength-dependent but direction-independent.

Structural dependence. It follows that only the canopy structure—the spatial distribution and orientation of foliage elements—determines the extinction of radiation in vegetation canopies. Radiative transfer in vegetation is therefore fundamentally unlike standard radiative transfer in atmospheres, water bodies, snowpacks, and similar media.

Inapplicability of classical methods. Because of the directional dependence of σ, the standard methods of solving the radiative transfer equation developed in astrophysics and atmospheric physics—which assume a direction-independent extinction coefficient—cannot in general be applied to radiative transfer in vegetation media.

Observational consequence. The extinction coefficient $\sigma(\mathbf{r}, \mathbf{\Omega})$ can be estimated from transmission measurements made below the canopy at wavelengths where leaves strongly absorb the incident radiation. Such measurements can, in principle, be inverted to recover the leaf area density $u_L(\mathbf{r})$ and the leaf-normal orientation distribution $g_L(\mathbf{r}, \mathbf{\Omega}_L)$.

§7. Limitations.

The derivation of the extinction coefficient presented above rests upon assumptions that are, strictly speaking, contradictory.

On the one hand, we require that the elementary volume contain a sufficient number of leaf elements for the leaf area density and the leaf-normal distribution to be realized as continuous and differentiable functions. On the other hand, we assume that the projections of these elements do not overlap. If the volume contains enough elements to justify a continuous description, it becomes increasingly improbable that their shadows—which have finite extent—would not overlap.

In reality, there is considerable mutual shading between leaves in a canopy. Our formulation, being probabilistic, does not account for the finite size of leaves and the consequent overlap of their shadows. As we shall see in a later chapter, the finite dimensions of leaves give rise to correlations in photon mean free paths between directions close to one another, producing a brightening of reflectance about the retro-solar direction known as the hot spot. The present formulation does not capture this correlation, and the hot-spot effect is therefore not represented.

These limitations notwithstanding, the turbid-medium model has proved remarkably successful in a wide range of applications, and the extinction coefficient as derived here remains the standard starting point for vegetation radiative transfer theory.

§8. Analytical forms of the G-function.

Explicit analytical expressions for the G-function can be obtained for the classical leaf-normal inclination distributions. We assume throughout that the azimuthal distribution of leaf normals is uniform, so that $h_L(\varphi_L) = 1$ and the G-function depends only upon the beam zenith angle θ. The extension to non-uniform azimuthal distributions is treated in Appendix C.

The azimuthal projection integral. As shown in Appendix A, §5–§8, the G-function for a canopy with azimuthally uniform leaf normals may be written as

$$G(\theta) = \int_0^{\pi/2} \overline{g}_L\,(\theta_L)\,\psi(\mu,\mu_L)\,\sin\theta_L\,d\theta_L,$$

where $\mu = \cos\theta$, $\mu_L = \cos\theta_L$, and $\psi(\mu,\mu_L)$ is the azimuthal projection integral—the average of the absolute direction cosine $|\mathbf{\Omega}_L \cdot \mathbf{\Omega}|$ over the leaf azimuth. The leaf-normal inclination distribution $\overline{g}_L(\theta_L)$ is defined as a probability density function with respect to the solid-angle measure $\sin\theta_L\,d\theta_L$ on the upper hemisphere, so that

$$\int_0^{\pi/2} \overline{g}_L\,(\theta_L)\,\sin\theta_L\,d\theta_L = 1.$$

The evaluation of ψ yields the piecewise result (Appendix A, §7–§8):

$$\psi(\mu,\mu_L) = \begin{cases} |\mu\mu_L|, & \text{if } |\mu\mu_L| \geq \sin\theta\,\sin\theta_L, \\ \mu\mu_L\left(\dfrac{2\varphi_t}{\pi} - 1\right) + \dfrac{2}{\pi}\sin\theta\,\sin\theta_L\,\sin\varphi_t, & \text{otherwise,} \end{cases}$$

where $\varphi_t = \arccos(-\mu\mu_L/\sin\theta\,\sin\theta_L)$.

Special cases: single leaf inclination. When all leaves share a common inclination θ_0, the distribution reduces to $\overline{g}_L(\theta_L) = \delta(\theta_L - \theta_0)/\sin\theta_0$, and the sifting property of the delta function gives $G(\theta) = \psi(\mu, \cos\theta_0)$ directly (Appendix A, §9–§11). For horizontal leaves ($\theta_0 = 0$) we recover $G = \cos\theta$ (Appendix A, §12), and for vertical leaves ($\theta_0 = \pi/2$) we obtain $G = (2/\pi)\sin\theta$ (Appendix A, §13).

The standard trigonometric distributions. For general distributions, the evaluation proceeds by expressing $\overline{g}_L$ as a linear combination of 1, $\cos 2\theta_L$, and $\cos 4\theta_L$ (via the standard double-angle identities) and exploiting the linearity of the integral. As established in Appendix A, §16–§19, the G-function then decomposes as $G = c_0\,\mathcal{J}_0 + c_2\,\mathcal{J}_2 + c_4\,\mathcal{J}_4$, where the fundamental integrals take the closed forms

$$\mathcal{J}_0 = \frac{1}{2}, \qquad \mathcal{J}_2 = -\frac{\sin^2\theta}{4}, \qquad \mathcal{J}_4 = -\frac{\cos^4\theta}{6}.$$

Substituting the expansion coefficients for each distribution (Appendix A, §20) yields the following results.

Planophile ($\overline{g}_L = 3\cos^2\theta_L$; predominantly horizontal leaves):

$$G(\theta) = \frac{3(1+\cos^2\theta)}{8}.$$

This function is large for small θ (near-vertical illumination) and decreases as the beam becomes more oblique, reflecting the fact that horizontal leaves present their maximum projected area to a vertical beam.

Erectophile ($\overline{g}_L = \frac{3}{2}\sin^2\theta_L$; predominantly vertical leaves):

$$G(\theta) = \frac{3(2+\sin^2\theta)}{16}.$$

The behaviour is the converse of the planophile case: G is smaller for near-vertical beams and increases toward horizontal incidence.

Plagiophile ($\overline{g}_L = \frac{15}{8}\sin^2 2\theta_L$; leaves concentrated near 45°):

$$G(\theta) = \frac{5(3+\cos^4\theta)}{32}.$$

The G-function varies modestly with beam zenith angle, reflecting the intermediate character of this distribution.

Extremophile ($\overline{g}_L = \frac{15}{7}\cos^2 2\theta_L$; bimodal—horizontal and vertical):

$$G(\theta) = \frac{5(3-\cos^4\theta)}{28}.$$

Uniform distribution ($\overline{g}_L = 1$; all inclination angles equally probable):

$$G = \frac{1}{2},$$

independent of θ, consistent with the result of Appendix A, §2–§4.

Each of these G-functions (Figure 4) has been verified to satisfy the normalization condition $\int_0^{\pi/2} G\,(\theta)\sin\theta\, d\theta = 1/2$, as proved in general by the master identity of Appendix B, §22–§23 and confirmed case by case in Appendix B, §25–§32. The summary table of Appendix B, §33 collects all results.

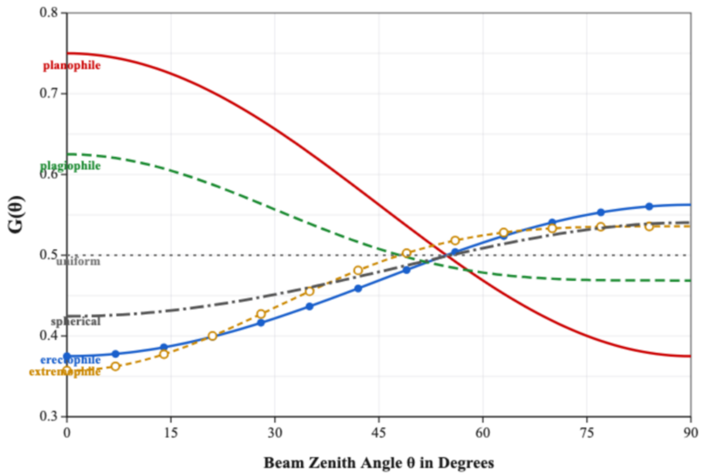

Figure 4. The geometry function G for various models of leaf normal oriientation distributions.

Non-uniform azimuthal distributions. When the leaf azimuthal distribution is not uniform but takes the form $h_L(\varphi_L) = 2\cos^2(\varphi - \varphi_L - \eta)$, the azimuthal projection integral acquires an additional correction term. As derived in Appendix C, §40–§45, the result is

$$\psi_{NU}(\mu, \mu_L, \eta) = \frac{1}{2}\,\psi_U(\mu, \mu_L) + \Delta\psi(\mu, \mu_L, \eta),$$

where ψ_U is the uniform result and the correction $\Delta\psi$ vanishes whenever $|\mu\mu_L| \geq \sin\theta \sin\theta_L$ (Case I of Appendix C, §44). The non-uniform azimuthal preference thus makes no difference when the sun is nearly vertical or the leaves nearly horizontal; the correction is largest when both θ and θ_L are far from zero and the canopy is illuminated obliquely (Appendix C, §47).

§9. The G-function in coniferous canopies.

In coniferous canopies, individual needles are small, cylindrical, and densely packed along shoots. The mutual shading of needles within a single shoot causes the effective projection area of the shoot to be substantially less than the sum of the individual needle projections. The turbid-medium formulation developed in the preceding sections, which treats each foliage element as an independent planar scatterer, is therefore inadequate at the needle level.

The resolution, introduced by Oker-Blom and Smolander (1988) and developed by Stenberg and collaborators in a series of papers, is to treat the shoot—the twig together with its complement of needles—as the basic scattering element, replacing the individual needle. The central quantity in this shoot-level formulation is the Silhouette to Total needle Area Ratio, abbreviated STAR.

For a given shoot viewed from a direction $\mathbf{\Omega}$, STAR is defined as

$$\text{STAR}(\mathbf{\Omega}) = \frac{S(\mathbf{\Omega})}{A_t},$$

where $S(\mathbf{\Omega})$ is the silhouette area of the shoot as seen from direction $\mathbf{\Omega}$, and A_t is the total (or hemi-surface, i.e., half the total surface) area of all needles on the shoot.

The spherically averaged STAR is obtained by averaging over all directions uniformly:

$$\overline{\mathrm{STAR}} = \frac{1}{4\pi} \int_{4\pi} \mathrm{STAR}\,(\mathbf{\Omega})\, d\mathbf{\Omega}.$$

For a single convex body—such as an isolated cylindrical needle—the mean projection averaged over all directions equals exactly one-quarter of the total surface area. Thus, for a shoot with no within-shoot shading,

$$\overline{\mathrm{STAR}}_{\text{no shading}} = 0.25.$$

Any reduction below this value is attributable to the mutual shading of needles within the shoot. Oker-Blom and Smolander (1988) measured $\overline{\mathrm{STAR}}$ for 258 Scots pine shoots and found values ranging from 0.09 to 0.21, with a mean of 0.14—representing a 43% reduction in light interception efficiency due to within-shoot shading. In the absence of species-specific data, $\overline{\mathrm{STAR}} \approx 0.16$ provides a reasonable approximation for a wide range of shoot structures.

§10. Concluding remarks.

We have derived the extinction coefficient for a vegetation canopy from first principles, proceeding from the probability of photon–leaf interaction in an elementary volume to the factored form $\sigma = u_L\, G$. The G-function encodes the geometric properties of the foliage, is independent of wavelength, and satisfies the normalization $\langle G \rangle = 1/2$ when averaged over all directions. Explicit analytical forms were given for the classical leaf-angle distributions (with complete derivations in Appendix A and normalization proofs in Appendix B), and the extension to non-uniform azimuthal distributions was treated in Appendix C. The adaptation required for coniferous canopies—replacing the leaf with the shoot and G with STAR—was presented in §9.

The extinction coefficient as derived here enters directly into the integro-differential and integral formulations of the radiative transfer equation for vegetation canopies. Its determination is therefore the essential first step in any quantitative treatment of radiation propagation through vegetated media.

References.

1. Ross, J. 1981. *The radiation regime and architecture of plant stands*. Boston: Dr. W. Junk Publishers.
2. Oker-Blom, P., & Smolander, H. (1988). The ratio of shoot silhouette area to total needle area in Scots pine. *Forest Science*, 34(4), 894-906.

Appendix A. Derivations of the G-Function

§1. Introduction. In this appendix we derive closed-form expressions for the geometry function $G(\mathbf{\Omega})$, which describes the mean projection of foliage area elements in an arbitrary direction $\mathbf{\Omega}$. We begin with the simplest case — the uniform (isotropic) leaf-normal distribution — for which the result can be obtained by an elementary symmetry argument. We then derive the azimuthal projection integral $\psi(\mu, \mu_L)$, which serves as the foundation upon which all subsequent results rest. With ψ in hand, we treat the case of leaves inclined at a single fixed angle θ_0 (including the limiting cases of horizontal and vertical foliage), and finally we obtain closed-form solutions for the four standard trigonometric leaf-angle distributions.

Throughout we write $\mu = \cos\theta$ for the direction cosine of the beam, $\mu_L = \cos\theta_L$ for that of the leaf normal, and adopt the convention that angles are measured from the outward normal to the horizontal plane. The leaf-normal inclination distribution $\overline{g}_L(\theta_L)$ is defined as a probability density function with respect to the solid-angle measure $\sin\theta_L \, d\theta_L$ on the upper hemisphere, so that

$$\int_0^{\pi/2} \overline{g}_L(\theta_L) \sin\theta_L \, d\theta_L = 1 \,.$$

With azimuthally uniform leaf normals, $h_L(\varphi_L) = 1/(2\pi)$, the G-function takes the form

$$G(\theta) = \int_0^{\pi/2} \overline{g}_L(\theta_L) \, \psi(\mu, \mu_L) \sin\theta_L \, d\theta_L \,.$$

Part I. The G-Function for Uniform Leaf Normals

§2. Statement of the problem. The G-function is defined as

$$G(\mathbf{\Omega}) = \frac{1}{2\pi} \int_{2\pi^+} g_L(\mathbf{\Omega}_L) |\mathbf{\Omega}_L \cdot \mathbf{\Omega}| d\mathbf{\Omega}_L \,,$$

where the integration extends over the upper hemisphere of leaf-normal directions and $g_L(\mathbf{\Omega}_L)$ is the leaf-normal orientation density. We wish to show that when $\overline{g}_L = 1$ (isotropic distribution),

$$G(\mathbf{\Omega}) = 1/2 \,,$$

independent of the beam direction $\mathbf{\Omega}$.

§3. Derivation. The element of solid angle on the upper hemisphere is $d\mathbf{\Omega}_L = \sin\theta_L \, d\theta_L \, d\varphi_L$, and the dot product of the two unit vectors is

$$\mathbf{\Omega}_L \cdot \mathbf{\Omega} = \sin\theta_L \sin\theta \cos(\varphi_L - \varphi) + \cos\theta_L \cos\theta \,.$$

Setting $\overline{g}_L = 1$, the G-function becomes

$$G = \frac{1}{2\pi} \int_0^{2\pi} d\,\varphi_L \int_0^{\pi/2} |\mathbf{\Omega}_L \cdot \mathbf{\Omega}| \sin\theta_L \, d\theta_L \,.$$

Since the leaf-normal distribution is isotropic, we are at liberty to choose the coordinate system. Let the polar axis point along $\mathbf{\Omega}$ itself, so that $\mathbf{\Omega} = \hat{z}$ and $\theta = 0$. The dot product collapses to

$$\mathbf{\Omega}_L \cdot \mathbf{\Omega} = \cos\theta_L = \mu_L\,.$$

On the upper hemisphere $\theta_L \in [0, \pi/2]$, whence $\mu_L \geq 0$ and $|\mu_L| = \mu_L$. The integrand no longer depends on φ_L, so the azimuthal integral yields 2π, cancelling the prefactor:

$$G = \int_0^{\pi/2} \cos\theta_L \sin\theta_L \, d\theta_L\,.$$

Substituting $u = \sin\theta_L$:

$$G = \int_0^1 u \, du = \frac{1}{2}.$$

§4. Remark on generality. The choice $\mathbf{\Omega} = \hat{z}$ entails no loss of generality. When $\overline{g}_L = 1$ the distribution is invariant under rotations of the coordinate frame, and the integrand $|\mathbf{\Omega}_L \cdot \mathbf{\Omega}|$ depends only on the angle between the two vectors. The integration over all leaf normals therefore sees the same geometry regardless of the direction of $\mathbf{\Omega}$, and $G = \frac{1}{2}$ universally.

Part II. The Azimuthal Projection Integral

§5. Statement of the problem. We now evaluate the azimuthal average of the absolute direction cosine,

$$\psi(\mu, \mu_L) = \frac{1}{2\pi} \int_0^{2\pi} |\mathbf{\Omega}_L \cdot \mathbf{\Omega}| \, d\varphi_L\,,$$

and show that

$$\psi(\mu, \mu_L) = \begin{cases} |\mu\mu_L|, & \text{if } |\mu\mu_L| \geq \sin\theta\sin\theta_L, \\ \mu\mu_L\left(\dfrac{2\varphi_t}{\pi} - 1\right) + \dfrac{2}{\pi}\sqrt{1-\mu^2}\,\sqrt{1-\mu_L^2}\,\sin\varphi_t, & \text{otherwise,} \end{cases}$$

where $\varphi_t = \arccos(-\mu\mu_L/\sin\theta\sin\theta_L)$.

§6. Notation. We introduce the abbreviations

$$A \equiv \mu\mu_L\,, \qquad B \equiv \sqrt{1-\mu^2}\,\sqrt{1-\mu_L^2} = \sin\theta\sin\theta_L\,,$$

so that $B \geq 0$ and

$$\mathbf{\Omega}_L \cdot \mathbf{\Omega} = A + B\cos(\varphi_L - \varphi)\,.$$

Setting $\varphi_t = \varphi_L - \varphi$ (a shift which does not alter the value of an integral over a full period), we have

$$\psi = \frac{1}{2\pi}\int_0^{2\pi} |A + B\cos\varphi_t| \; d\varphi_t \,.$$

§7. Case 1: $|A| \geq B$. As φ_t ranges over $[0,2\pi]$, the expression $A + B\cos\varphi_t$ oscillates between $A - B$ and $A + B$. When $|A| \geq B$, these extreme values have the same sign, and the expression never vanishes. The absolute value may then be removed at the cost of an overall sign, and the evaluation proceeds directly:

$$\psi = \frac{1}{2\pi}\left|\int_0^{2\pi} (A + B\cos\varphi_t) \; d\varphi_t\right| = \frac{1}{2\pi}|2\pi A + 0| = |A| = |\mu\mu_L| \,,$$

where we have used $\int_0^{2\pi} \cos\varphi_t \, d\varphi_t = 0$.

§8. Case 2: $|A| < B$. The expression $A + B\cos\varphi_t$ now changes sign. Setting it equal to zero gives $\cos\varphi_t = -A/B$, which, since $|A/B| < 1$, has the two solutions $\varphi_t = \varphi_t^*$ and $\varphi_t = 2\pi - \varphi_t^*$, where

$$\varphi_t^* \equiv \arccos\left(-\frac{A}{B}\right).$$

For $\varphi_t \in [0, \varphi_t^*]$ the expression is positive; for $\varphi_t \in [\varphi_t^*, 2\pi - \varphi_t^*]$ it is negative; and for $\varphi_t \in [2\pi - \varphi_t^*, 2\pi]$ it is again positive. We split accordingly:

$$2\pi\psi = \int_0^{\varphi_t^*} (A + B\cos\varphi_t) \; d\varphi_t - \int_{\varphi_t^*}^{2\pi-\varphi_t^*} (A + B\cos\varphi_t) \; d\varphi_t + \int_{2\pi-\varphi_t^*}^{2\pi} (A + B\cos\varphi_t) \; d\varphi_t \,.$$

By the symmetry $\varphi_t \to 2\pi - \varphi_t$ the first and third integrals are equal. Writing

$$I_+ = 2\int_0^{\varphi_t^*} (A + B\cos\varphi_t) \; d\varphi_t \,, \qquad I_- = \int_{\varphi_t^*}^{2\pi-\varphi_t^*} (A + B\cos\varphi_t) \; d\varphi_t \,,$$

we have $2\pi\psi = I_+ - I_-$. But $I_+ + I_- = \int_0^{2\pi} (A + B\cos\varphi_t) \; d\varphi_t = 2\pi A$, so that $I_- = 2\pi A - I_+$ and

$$2\pi\psi = 2I_+ - 2\pi A \,.$$

Evaluating I_+:

$$I_+ = 2[A\varphi_t^* + B\sin\varphi_t^*] \,.$$

Hence

$$\psi = \frac{4A\varphi_t^* + 4B\sin\varphi_t^* - 2\pi A}{2\pi} = A\left(\frac{2\varphi_t^*}{\pi} - 1\right) + \frac{2B}{\pi}\sin\varphi_t^* \,.$$

Restoring the original variables:

$$\psi(\mu, \mu_L) = \mu\mu_L\left(\frac{2\varphi_t}{\pi} - 1\right) + \frac{2}{\pi}\sin\theta\sin\theta_L\sin\varphi_t \,,$$

where $\varphi_t = \arccos(-\mu\mu_L/\sin\theta\sin\theta_L)$. This completes the derivation.

Part III. The G-Function for a Single Leaf Inclination Angle

§9. Statement of the problem. We now take the leaf-angle distribution to be concentrated at a single inclination θ_0, setting $\overline{g}_L(\theta_L) = \delta(\theta_L - \theta_0)/\sin\theta_0$ (where the factor $1/\sin\theta_0$ ensures proper normalisation against the solid-angle measure $\sin\theta_L\, d\theta_L$). The G-function becomes

$$G(\mathbf{\Omega}) = \int_0^{\pi/2} \frac{\delta(\theta_L - \theta_0)}{\sin\theta_0}\, \psi(\mu, \mu_L) \sin\theta_L\, d\theta_L\,.$$

By the sifting property of the delta function, the factors $\sin\theta_L$ and $1/\sin\theta_0$ cancel, and we obtain simply

$$G(\theta) = \psi(\mu, \cos\theta_0)\,.$$

It remains to determine which branch of ψ applies.

§10. The branch condition. The condition $|\mu\mu_0| \geq \sin\theta\sin\theta_0$ (with $\mu_0 = \cos\theta_0$) is equivalent, for θ and θ_0 in $[0, \pi/2]$, to $\cos\theta\cos\theta_0 \geq \sin\theta\sin\theta_0$, which is to say $\cos(\theta + \theta_0) \geq 0$, or

$$\theta + \theta_0 \leq \frac{\pi}{2}.$$

§11. Result for general θ_0. Combining the two branches:

$$G(\theta, \theta_0) = \begin{cases} \cos\theta\cos\theta_0, & \text{if } \theta + \theta_0 \leq \pi/2, \\ \cos\theta\cos\theta_0 \left(\dfrac{2\varphi_t}{\pi} - 1\right) + \dfrac{2}{\pi}\sin\theta\sin\theta_0\sin\varphi_t, & \text{if } \theta + \theta_0 > \pi/2, \end{cases}$$

where $\varphi_t = \arccos(-\cot\theta\cot\theta_0)$.

§12. Horizontal leaves ($\theta_0 = 0$). With $\mu_0 = 1$ we have $\theta + 0 \leq \pi/2$ for all θ in the upper hemisphere, so the first branch always applies:

$$G = \cos\theta\,.$$

This is physically transparent: a horizontal leaf presents its full area to a vertical beam and none to a horizontal one.

§13. Vertical leaves ($\theta_0 = \pi/2$). With $\mu_0 = 0$ the condition $\theta + \pi/2 > \pi/2$ holds whenever $\theta > 0$, so the second branch applies. Since $\cot\theta_0 = 0$, we have $\varphi_t = \arccos(0) = \pi/2$, whence $\sin\varphi_t = 1$ and $2\varphi_t/\pi - 1 = 0$:

$$G = \frac{2}{\pi}\sin\theta\,.$$

A vertical beam ($\theta = 0$) sees no projected area, while a horizontal beam ($\theta = \pi/2$) sees a projection of $2/\pi \approx 0.637$ — less than unity because the leaves, though all vertical, are randomly oriented in azimuth.

§14. Verification. The general formula of §11 reduces to the horizontal and vertical results of §12 and §13 as special cases, providing an internal consistency check.

Part IV. The G-Function for Standard Leaf-Angle Distributions

§15. Statement of the problem. We evaluate $G(\mathbf{\Omega})$ for the four standard trigonometric distributions. Each is a probability density with respect to the solid-angle measure $\sin\theta_L\, d\theta_L$:

1. Planophile (predominantly horizontal): $\overline{g}_L(\theta_L) = 3\cos^2\theta_L$.

$$\int_0^{\pi/2} 3\cos^2\theta_L \sin\theta_L\, d\theta_L = 3\left[-\frac{\cos^3\theta_L}{3}\right]_0^{\pi/2} = 1. \quad ✓$$

1. Erectophile (predominantly vertical): $\overline{g}_L(\theta_L) = \frac{3}{2}\sin^2\theta_L$.

$$\int_0^{\pi/2} \frac{3}{2}\sin^2\theta_L \sin\theta_L\, d\theta_L = \frac{3}{2}\int_0^{\pi/2} \sin^3\theta_L\, d\theta_L = \frac{3}{2}\cdot\frac{2}{3} = 1. \quad ✓$$

1. Plagiophile (predominantly at 45°): $\overline{g}_L(\theta_L) = \frac{15}{8}\sin^2 2\theta_L$.

Writing $\sin^2 2\theta_L = 4\sin^2\theta_L\cos^2\theta_L$ and substituting $u = \cos\theta_L$:

$$\int_0^{\pi/2} \frac{15}{8}\cdot 4\sin^2\theta_L\cos^2\theta_L \sin\theta_L\, d\theta_L = \frac{15}{2}\int_0^{1} (1-u^2)\, u^2\, du = \frac{15}{2}\left[\frac{1}{3}-\frac{1}{5}\right] = 1. \quad ✓$$

1. Extremophile (bimodal — horizontal and vertical): $\overline{g}_L(\theta_L) = \frac{15}{7}\cos^2 2\theta_L$.

Writing $\cos^2 2\theta_L = (2\cos^2\theta_L - 1)^2 = 4\cos^4\theta_L - 4\cos^2\theta_L + 1$ and substituting $u = \cos\theta_L$:

$$\int_0^{\pi/2} \frac{15}{7}(4u^4 - 4u^2 + 1)\, du = \frac{15}{7}\left[\frac{4}{5}-\frac{4}{3}+1\right] = \frac{15}{7}\cdot\frac{7}{15} = 1. \quad ✓$$

§16. Strategy: linearity and the fundamental integrals. Each distribution can be expanded as a linear combination of 1, $\cos 2\theta_L$, and $\cos 4\theta_L$ by means of the standard double-angle identities $\cos^2\theta_L = \frac{1}{2}(1+\cos 2\theta_L)$, $\sin^2\theta_L = \frac{1}{2}(1-\cos 2\theta_L)$, $\sin^2 2\theta_L = \frac{1}{2}(1-\cos 4\theta_L)$, and $\cos^2 2\theta_L = \frac{1}{2}(1+\cos 4\theta_L)$. Each $\overline{g}_L$ thus takes the form $c_0 + c_2\cos 2\theta_L + c_4\cos 4\theta_L$, and by the linearity of integration the G-function decomposes as

$$G = c_0\,\mathcal{J}_0 + c_2\,\mathcal{J}_2 + c_4\,\mathcal{J}_4\,,$$

where we have defined the fundamental integrals

$$\mathcal{J}_0 \equiv \int_0^{\pi/2} \psi(\mu,\mu_L)\sin\theta_L\, d\theta_L\,, \qquad \mathcal{J}_n \equiv \int_0^{\pi/2} \cos(n\theta_L)\,\psi(\mu,\mu_L)\sin\theta_L\, d\theta_L\,.$$

If $\mathcal{J}_0$, $\mathcal{J}_2$, and $\mathcal{J}_4$ can be evaluated in closed form, all four G-functions follow at once.

§17. Evaluation of $\mathcal{J}_0$. By the coordinate-alignment argument of §3 (choose the polar axis along $\mathbf{\Omega}$ so that $\psi = \cos\theta_L$):

$$\mathcal{J}_0 = \int_0^{\pi/2} \cos\theta_L \sin\theta_L \, d\theta_L = \frac{1}{2}.$$

§18. Evaluation of $\mathcal{J}_2$. The branch condition splits the domain at $\theta_L^* = \pi/2 - \theta$. In the first branch, $\psi = \cos\theta\cos\theta_L$. Writing $\cos(2\theta_L)\cos\theta_L\sin\theta_L = \frac{1}{2}\cos(2\theta_L)\sin(2\theta_L) = \frac{1}{4}\sin(4\theta_L)$:

$$\mathcal{J}_2^{(1)} = \frac{\cos\theta}{4}\int_0^{\pi/2-\theta} \sin(4\theta_L)\, d\theta_L = \frac{\cos\theta}{16}[1 - \cos(2\pi - 4\theta)] = \frac{\cos\theta}{16} \cdot 8\sin^2\theta\cos^2\theta = \frac{\sin^2\theta\cos^3\theta}{2}.$$

The second branch, though more involved, can be evaluated by the substitution $u = \cot\theta_L/\tan\theta$ and integration by parts. Combining both branches, the result takes the remarkably simple form

$$\mathcal{J}_2 = -\frac{\sin^2\theta}{4}.$$

One may verify this at the boundary values: $\mathcal{J}_2 = 0$ when $\theta = 0$, and $\mathcal{J}_2 = -\frac{1}{4}$ when $\theta = \pi/2$.

§19. Evaluation of $\mathcal{J}_4$. By an analogous calculation one obtains

$$\mathcal{J}_4 = -\frac{\cos^4\theta}{6},$$

which vanishes at $\theta = \pi/2$ and equals $-\frac{1}{6}$ at $\theta = 0$.

§20. The closed-form G-functions. We now expand each distribution and substitute the fundamental integrals.

Planophile. $\overline{g}_L = 3\cos^2\theta_L = \frac{3}{2}(1 + \cos 2\theta_L)$, giving $c_0 = 3/2$, $c_2 = 3/2$, $c_4 = 0$:

$$G = \frac{3}{2}\cdot\frac{1}{2} + \frac{3}{2}\left(-\frac{\sin^2\theta}{4}\right) = \frac{3}{4} - \frac{3\sin^2\theta}{8} = \frac{3(1+\cos^2\theta)}{8}.$$

Erectophile. $\overline{g}_L = \frac{3}{2}\sin^2\theta_L = \frac{3}{4}(1 - \cos 2\theta_L)$, giving $c_0 = 3/4$, $c_2 = -3/4$, $c_4 = 0$:

$$G = \frac{3}{4}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{\sin^2\theta}{4} = \frac{3}{8} + \frac{3\sin^2\theta}{16} = \frac{3(2+\sin^2\theta)}{16}.$$

Plagiophile. $\overline{g}_L = \frac{15}{8}\sin^2 2\theta_L = \frac{15}{16}(1 - \cos 4\theta_L)$, giving $c_0 = 15/16$, $c_2 = 0$, $c_4 = -15/16$:

$$G = \frac{15}{16}\cdot\frac{1}{2} + \frac{15}{16}\cdot\frac{\cos^4\theta}{6} = \frac{15}{32} + \frac{15\cos^4\theta}{96} = \frac{5(3+\cos^4\theta)}{32}.$$

Extremophile. $\overline{g}_L = \frac{15}{7}\cos^2 2\theta_L = \frac{15}{14}(1+\cos 4\theta_L)$, giving $c_0 = 15/14$, $c_2 = 0$, $c_4 = 15/14$:

$$G = \frac{15}{14}\cdot\frac{1}{2} + \frac{15}{14}\left(-\frac{\cos^4\theta}{6}\right) = \frac{15}{28} - \frac{15\cos^4\theta}{84} = \frac{5(3-\cos^4\theta)}{28}.$$

Appendix B. Verification of the Normalisation Condition

§21. Introduction. In this appendix we prove that each G-function derived in Appendix A satisfies the normalisation condition

$$\frac{1}{2\pi}\int_{2\pi} G\,(\mathbf{\Omega})\,d\mathbf{\Omega} = \frac{1}{2}.$$

Since G depends only on the polar angle θ, this reduces to

$$\int_0^{\pi/2} G\,(\theta)\sin\theta\,d\theta = \frac{1}{2}.$$

We first establish a master identity from which all cases follow, then verify each G-function both through the identity and by direct integration.

Part I. The Master Identity

§22. Theorem. For any fixed $\theta_L \in [0, \pi/2]$,

$$\int_0^{\pi/2} \psi\,(\mu, \mu_L)\sin\theta\,d\theta = \frac{1}{2}.$$

§23. Proof. The integral represents the average of $|\mathbf{\Omega}_L \cdot \mathbf{\Omega}|$ over all beam directions $\mathbf{\Omega}$ in the upper hemisphere, for a fixed leaf-normal direction $\mathbf{\Omega}_L$. Place the polar axis along $\mathbf{\Omega}_L$, so that $\mathbf{\Omega}_L = \hat{z}$. In this frame the dot product becomes $\mathbf{\Omega}_L \cdot \mathbf{\Omega} = \cos\alpha$, where α is the angle between the two vectors. On the upper hemisphere $\alpha \in [0, \pi/2]$, whence $\cos\alpha \geq 0$ and the absolute value is trivial. The integral becomes

$$\int_0^{\pi/2} |\cos\alpha|\,\sin\alpha\,d\alpha = \int_0^{\pi/2} \cos\alpha\sin\alpha\,d\alpha = \left[\frac{\sin^2\alpha}{2}\right]_0^{\pi/2} = \frac{1}{2}.$$

Since the result is independent of the orientation of $\mathbf{\Omega}_L$ (we have integrated over all beam directions), it holds for every θ_L. ■

§24. Consequence. The G-function for a general distribution $\overline{g}_L(\theta_L)$ is $G(\theta) = \int_0^{\pi/2} \overline{g}_L\,\psi\sin\theta_L\,d\theta_L$. Integrating over beam directions and interchanging the order of integration (Fubini):

$$\int_0^{\pi/2} G(\theta)\sin\theta\, d\theta = \int_0^{\pi/2} \overline{g}_L(\theta_L)\sin\theta_L \left[\underbrace{\int_0^{\pi/2} \psi \sin\theta\, d\theta}_{=1/2}\right] d\theta_L = \frac{1}{2}\int_0^{\pi/2} \overline{g}_L(\theta_L)\sin\theta_L\, d\theta_L\,.$$

The normalisation of G therefore reduces to the normalisation of $\overline{g}_L$: the condition $\int_0^{\pi/2} G \sin\theta\, d\theta = \frac{1}{2}$ holds if and only if $\int_0^{\pi/2} \overline{g}_L \sin\theta_L\, d\theta_L = 1$. Since every distribution defined in Appendix A satisfies this normalisation, the G-function normalisation is guaranteed in all cases.

Part II. The Delta-Function Cases

§25. Uniform (isotropic) distribution ($\overline{g}_L = 1$, $G = \frac{1}{2}$).

$$\int_0^{\pi/2} \frac{1}{2}\sin\theta\, d\theta = \frac{1}{2}[-\cos\theta]_0^{\pi/2} = \frac{1}{2}\,. \quad ✓$$

§26. Horizontal leaves ($\theta_0 = 0$, $G = \cos\theta$).

$$\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta = \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}\,. \quad ✓$$

§27. Vertical leaves ($\theta_0 = \pi/2$, $G = (2/\pi)\sin\theta$).

$$\frac{2}{\pi}\int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2}\,. \quad ✓$$

§28. General constant inclination θ_0 ($G = \psi(\mu, \cos\theta_0)$).

By the master identity of §22, $\int_0^{\pi/2} \psi(\mu, \mu_L)\sin\theta\, d\theta = \frac{1}{2}$ for any μ_L. Setting $\mu_L = \cos\theta_0$ gives

$$\int_0^{\pi/2} G(\theta)\sin\theta\, d\theta = \frac{1}{2}\,. \quad ✓$$

This subsumes the horizontal and vertical cases as the special instances $\theta_0 = 0$ and $\theta_0 = \pi/2$.

Part III. The Standard Distributions

For the four trigonometric distributions, the normalisation $\int_0^{\pi/2} \overline{g}_L \sin\theta_L\, d\theta_L = 1$ was established in §15 of Appendix A. By the master reduction of §24, the G-function normalisation $\int_0^{\pi/2} G \sin\theta\, d\theta = \frac{1}{2}$ follows at once. We nevertheless confirm each case by direct integration.

§29. Planophile ($\overline{g}_L = 3\cos^2\theta_L$, $G = 3(1 + \cos^2\theta)/8$).

$$\frac{3}{8}\int_0^{\pi/2} (1 + \cos^2\theta)\sin\theta\, d\theta = \frac{3}{8}\left[1 + \frac{1}{3}\right] = \frac{3}{8}\cdot\frac{4}{3} = \frac{1}{2}\,. \quad ✓$$

§30. Erectophile ($\overline{g}_L = \frac{3}{2}\sin^2\theta_L$, $G = 3(2+\sin^2\theta)/16$).

$$\frac{3}{16}\int_0^{\pi/2}(2+\sin^2\theta)\sin\theta\, d\theta = \frac{3}{16}\left[2+\frac{2}{3}\right] = \frac{3}{16}\cdot\frac{8}{3} = \frac{1}{2}. \quad ✓$$

§31. Plagiophile ($\overline{g}_L = \frac{15}{8}\sin^2 2\theta_L$, $G = 5(3+\cos^4\theta)/32$).

$$\frac{5}{32}\int_0^{\pi/2}(3+\cos^4\theta)\sin\theta\, d\theta = \frac{5}{32}\left[3+\frac{1}{5}\right] = \frac{5}{32}\cdot\frac{16}{5} = \frac{1}{2}. \quad ✓$$

§32. Extremophile ($\overline{g}_L = \frac{15}{7}\cos^2 2\theta_L$, $G = 5(3-\cos^4\theta)/28$).

$$\frac{5}{28}\int_0^{\pi/2}(3-\cos^4\theta)\sin\theta\, d\theta = \frac{5}{28}\left[3-\frac{1}{5}\right] = \frac{5}{28}\cdot\frac{14}{5} = \frac{1}{2}. \quad ✓$$

Part IV. Summary of the Normalisation Results

§33. The master identity $\int_0^{\pi/2}\psi\sin\theta\, d\theta = \frac{1}{2}$, proved in §22–23, reduces the normalisation of G to the normalisation of $\overline{g}_L$. Since every distribution satisfies $\int_0^{\pi/2}\overline{g}_L\sin\theta_L\, d\theta_L = 1$, the condition $\int_0^{\pi/2} G\sin\theta\, d\theta = \frac{1}{2}$ holds in all cases. The following table collects the results:

Case	$\overline{g}_L(\theta_L)$	$G(\theta)$	$\int_0^{\pi/2} G\sin\theta\, d\theta$
Uniform	1	$1/2$	$1/2$
Constant θ_0	$\delta(\theta_L-\theta_0)/\sin\theta_0$	$\psi(\mu,\cos\theta_0)$	$1/2$
Planophile	$3\cos^2\theta_L$	$3(1+\cos^2\theta)/8$	$1/2$
Erectophile	$\frac{3}{2}\sin^2\theta_L$	$3(2+\sin^2\theta)/16$	$1/2$
Plagiophile	$\frac{15}{8}\sin^2 2\theta_L$	$5(3+\cos^4\theta)/32$	$1/2$
Extremophile	$\frac{15}{7}\cos^2 2\theta_L$	$5(3-\cos^4\theta)/28$	$1/2$

Appendix C. The Azimuthal Projection Integral for Non-Uniform Leaf Orientations

§34. Introduction. In Appendices A and B we treated the geometry function $G(\mathbf{\Omega})$ and its normalisation under the assumption that the leaf azimuthal distribution is *uniform*, that is, $h_L(\varphi_L) = 1$. In this appendix we remove that restriction. We first establish the normalisation of a particular non-uniform azimuthal probability density h_L, and then derive the form of the azimuthal projection integral ψ that results when h_L is no longer constant. The analysis follows the method of Appendix A, §3–§5, but the azimuthal weighting introduces an additional trigonometric factor whose consequences we trace in detail.

§35. The non-uniform azimuthal PDF. Let the azimuthal distribution of leaf normals be given by

$$\frac{1}{2\pi}\, h_L(\varphi_L, \varphi) = \frac{1}{\pi}\cos^2(\varphi - \varphi_L - \eta),$$

where φ is the azimuth of the beam direction $\mathbf{\Omega}$ and η is a parameter specifying the preferred leaf azimuth. When $\eta = 0$ the distribution peaks for leaves whose normal lies in the plane containing $\mathbf{\Omega}$; as η varies, the axis of preferred orientation rotates.

§36. Isolation of h_L. From the definition in §35 we may isolate h_L itself by multiplying both sides by 2π:

$$h_L(\varphi_L) = 2\cos^2(\varphi - \varphi_L - \eta).$$

This is the function whose properties we must now establish.

§37. Normalisation — statement. We require that the azimuthal PDF satisfy the normalisation condition

$$\frac{1}{2\pi}\int_0^{2\pi} h_L\,(\varphi_L)\, d\varphi_L = 1.$$

§38. Normalisation — proof. Substituting the expression obtained in §36 into the integral of §37, we write

$$\frac{1}{2\pi}\int_0^{2\pi} 2\cos^2(\varphi - \varphi_L - \eta)\, d\varphi_L.$$

To evaluate this we apply the standard double-angle identity

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2},$$

with $\alpha = \varphi - \varphi_L - \eta$. The integral becomes

$$\frac{2}{2\pi}\int_0^{2\pi} \frac{1 + \cos 2\alpha}{2}\, d\varphi_L = \frac{1}{2\pi}\left[\int_0^{2\pi} d\,\varphi_L + \int_0^{2\pi} \cos 2\alpha\, d\varphi_L\right].$$

The first integral is elementary:

$$\int_0^{2\pi} d\,\varphi_L = 2\pi.$$

For the second we note that the cosine completes an integral number of full periods over the interval $[0,2\pi]$, so that

$$\int_0^{2\pi} \cos 2\alpha\, d\varphi_L = 0.$$

Combining, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} h_L\ d\varphi_L = \frac{1}{2\pi}[2\pi + 0] = 1,$$

which was to be proved. The factor of 2 in $h_L = 2\cos^2(\cdot)$ is thus precisely what is required: the mean value of $\cos^2$ over a full period is $1/2$, and the prefactor compensates this to yield unity. □

§39. Recapitulation of the uniform result. Before proceeding to the non-uniform case, we recall from Appendix A, §5 that with $h_L = 1$ (uniform azimuthal distribution) the azimuthal projection integral is defined by

$$\psi_U(\mu,\mu_L) = \frac{1}{2\pi}\int_0^{2\pi} |\mathbf{\Omega}_L \cdot \mathbf{\Omega}|\ d\varphi_L,$$

where the direction cosine is

$$\mathbf{\Omega}_L \cdot \mathbf{\Omega} = \mu\mu_L + \sin\theta\,\sin\theta_L\cos(\varphi_L - \varphi),$$

and the result is the piecewise expression: when $|\mu\mu_L| \geq \sin\theta\,\sin\theta_L$,

$$\psi_U = |\mu\mu_L|;$$

otherwise,

$$\psi_U = \mu\mu_L\left(\frac{2\varphi_t}{\pi} - 1\right) + \frac{2}{\pi}\sin\theta\,\sin\theta_L\,\sin\varphi_t,$$

with

$$\varphi_t = \arccos\left(\frac{-\mu\mu_L}{\sin\theta\,\sin\theta_L}\right).$$

§40. The non-uniform projection integral — formulation. We now consider the integral in which the azimuthal average is weighted by the non-uniform PDF:

$$\psi_{NU}(\mu,\mu_L) = \frac{1}{2\pi}\int_0^{2\pi} h_L\,(\varphi_L,\varphi)\,|\mathbf{\Omega}_L \cdot \mathbf{\Omega}|\,d\varphi_L.$$

Substituting $h_L = 2\cos^2(\varphi - \varphi_L - \eta)$ and the expression for the direction cosine from §39, and writing $a = \mu\mu_L$ and $b = \sin\theta\,\sin\theta_L$ for brevity:

$$\psi_{NU} = \frac{1}{2\pi}\int_0^{2\pi} 2\cos^2(\varphi - \varphi_L - \eta)\,|a + b\cos(\varphi_L - \varphi)|\,d\varphi_L.$$

§41. Change of variable. We introduce the substitution

$$\alpha = \varphi_L - \varphi, \qquad d\alpha = d\varphi_L.$$

Since the integrand is periodic with period 2π, the limits of integration remain 0 to 2π. Noting that $\cos^2(\varphi - \varphi_L - \eta) = \cos^2(\alpha + \eta)$, the integral becomes

$$\psi_{NU} = \frac{1}{\pi}\int_0^{2\pi} \cos^2(\alpha + \eta)\,|a + b\cos\alpha|\,d\alpha.$$

§42. Decomposition into uniform part and correction. The key to evaluating the integral in §41 is the identity

$$\cos^2(\alpha + \eta) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha + 2\eta).$$

Substituting into the expression for ψ_{NU} and distributing:

$$\psi_{NU} = \frac{1}{2\pi}\int_0^{2\pi} |a + b\cos\alpha|\,d\alpha + \frac{1}{2\pi}\int_0^{2\pi} \cos(2\alpha + 2\eta)\,|a + b\cos\alpha|\,d\alpha.$$

We recognise the first integral as exactly the *uniform* result ψ_U from §39. We thus write

$$\psi_{NU}(\mu, \mu_L, \eta) = \frac{1}{2}\,\psi_U(\mu, \mu_L) + \Delta\psi(\mu, \mu_L, \eta),$$

where the correction term is

$$\Delta\psi = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\alpha + 2\eta)\,|a + b\cos\alpha|\,d\alpha.$$

§43. Structure of the correction integral. The correction $\Delta\psi$ is an integral of the product of $\cos(2\alpha + 2\eta)$ with the same absolute-value function that appears in the uniform case. The integration domain is partitioned in exactly the same manner as in Appendix A, §5: the argument of the absolute value vanishes when

$$\cos\alpha = -a/b,$$

which defines the critical angle φ_t of §39. In the region where $|a| \geq b$, the expression inside the absolute value does not change sign and the absolute value may be removed directly. In the complementary region, the domain splits at $\alpha = \varphi_t$ and $\alpha = 2\pi - \varphi_t$.

§44. Evaluation — Case I: $|a| \geq b$. In this case $a + b\cos\alpha$ does not change sign on $[0,2\pi]$, so the absolute value may be dropped. We write

$$\varDelta\psi = \frac{1}{2\pi}\int_0^{2\pi} \cos\,(2\alpha + 2\eta)\,(a + b\cos\alpha)\,d\alpha.$$

Expanding the product:

$$\varDelta\psi = \frac{a}{2\pi}\int_0^{2\pi} \cos\,(2\alpha + 2\eta)\,d\alpha + \frac{b}{2\pi}\int_0^{2\pi} \cos\,(2\alpha + 2\eta)\cos\alpha\,d\alpha.$$

The first integral vanishes since $\cos(2\alpha + 2\eta)$ completes two full periods over $[0,2\pi]$:

$$\int_0^{2\pi} \cos\,(2\alpha + 2\eta)\,d\alpha = 0.$$

For the second we use $\cos A\cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$ with $A = 2\alpha + 2\eta$, $B = \alpha$:

$$\cos(2\alpha + 2\eta)\cos\alpha = \frac{1}{2}[\cos(\alpha + 2\eta) + \cos(3\alpha + 2\eta)].$$

Both $\cos(\alpha + 2\eta)$ and $\cos(3\alpha + 2\eta)$ integrate to zero over $[0,2\pi]$. Therefore

$$\varDelta\psi = 0 \qquad \text{(Case I)}.$$

This is a notable result: when the sun is nearly vertical or the leaves nearly horizontal (so that $|a| \geq b$), the non-uniform azimuthal preference makes no difference, and

$$\psi_{NU} = 1/2\ \psi_U = 1/2\ |a|.$$

§45. Evaluation — Case II: $|a| < b$. In this case $a + b\cos\alpha$ changes sign. The domain splits into three sub-intervals:

(i) $\alpha \in [0, \varphi_t]$, where $a + b\cos\alpha > 0$,

(ii) $\alpha \in [\varphi_t, 2\pi - \varphi_t]$, where $a + b\cos\alpha < 0$,

(iii) $\alpha \in [2\pi - \varphi_t, 2\pi]$, where $a + b\cos\alpha > 0$.

Resolving the absolute value in each sub-interval and combining (i) and (iii) by periodicity, the primitives needed are:

$$\int \cos(2\alpha + 2\eta)\,d\alpha = 1/2\sin(2\alpha + 2\eta),$$

$$\int \cos(2\alpha + 2\eta)\cos\alpha\,d\alpha = 1/2\sin(\alpha + 2\eta) + 1/6\sin(3\alpha + 2\eta),$$

where the second follows from the product-to-sum formula. Evaluating at the boundary values and simplifying, the correction is

$$\varDelta\psi = \frac{a}{2\pi}[2\sin(2\varphi_t + 2\eta) - \sin 4\eta]$$

$$+\frac{b}{2\pi}[\sin(\varphi_t + 2\eta) + 1/3\sin(3\varphi_t + 2\eta) - 4/3\sin 2\eta],$$

where $a = \mu\mu_L$, $b = \sin\theta\,\sin\theta_L$, and $\varphi_t = \arccos(-a/b)$.

§46. Summary. Collecting the results of §§42–45, the azimuthal projection integral for the non-uniform leaf azimuthal distribution $h_L = 2\cos^2(\varphi - \varphi_L - \eta)$ is

$$\psi_{NU}(\mu, \mu_L, \eta) = 1/2\ \psi_U(\mu, \mu_L) + \Delta\psi(\mu, \mu_L, \eta),$$

where ψ_U is the uniform result of Appendix A, §5. The correction is

$$\Delta\psi = 0, \qquad \text{if } |\mu\mu_L| \geq \sin\theta\,\sin\theta_L;$$

otherwise

$$\Delta\psi = \frac{a}{2\pi}[2\sin(2\varphi_t + 2\eta) - \sin 4\eta] + \frac{b}{2\pi}[\sin(\varphi_t + 2\eta) + 1/3\sin(3\varphi_t + 2\eta) - 4/3\sin 2\eta].$$

§47. Limiting cases and interpretation. Several limiting cases serve as checks on the result.

(a) *Uniform limit.* If we average $\Delta\psi$ over all preferred orientations $\eta \in [0,2\pi]$, every trigonometric term in η averages to zero, so $\langle\Delta\psi\rangle_\eta = 0$ and $\langle\psi_{NU}\rangle_\eta = \frac{1}{2}\psi_U$. Since averaging h_L over η recovers the uniform case $h_L = 1$, the result is consistent.

(b) *Vertical sun ($\mu = 1$).* When $\theta = 0$ we have $\sin\theta = 0$, so $|a| = |\mu_L| \geq 0 = b$, and Case I applies. The projection integral reduces to $\psi_{NU} = \frac{1}{2}|\mu_L|$, independent of η.

(c) *Horizontal leaves ($\mu_L = 1$).* Similarly, when $\theta_L = 0$ we have $\sin\theta_L = 0$, and again Case I applies: $\psi_{NU} = \frac{1}{2}|\mu|$, independent of η.

(d) *Physical interpretation.* The parameter η encodes the azimuthal anisotropy of the canopy. When $\eta = 0$, leaves preferentially face the beam; when $\eta = \pi/2$, they face perpendicular to it. The correction $\Delta\psi$ is largest when both θ and θ_L are far from zero and the sun illuminates the canopy obliquely, precisely the regime in which the azimuthal arrangement of leaves matters most for radiative transfer.