

## EE 445/645: Physical Models in Remote Sensing (Spring 2026)

### Chapter 01-Part 07: The Principle of Reciprocity – Notes (See C1-P7 PPTs)

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#### Problem Statement

Consider a scattering and absorbing medium with:

- No internal emission ( $j = 0$ )
- Absorption coefficient:  $\sigma_a$
- Scattering coefficient:  $\sigma_s'$
- Total extinction coefficient:  $\sigma = \sigma_a + \sigma_s'$
- Phase function:  $P(\Omega' \rightarrow \Omega)$  describing scattering from direction  $\Omega'$  to direction  $\Omega$

**Goal:** Prove that **switching** the **source direction** and **detector direction** yields the same detected radiance.

#### Step 1: Radiative Transfer Equation with Scattering

The radiative transfer equation for a scattering medium without emission is:

$$\Omega \cdot \nabla I(r, \Omega) = -\sigma I(r, \Omega) + \sigma_s' \int_{4\pi} P(\Omega' \rightarrow \Omega) I(r, \Omega') d\Omega'$$

Where:

- $I(\mathbf{r}, \Omega)$  = radiance at position  $\mathbf{r}$  in direction  $\Omega$
- $\Omega \cdot \nabla I$  = directional derivative along  $\Omega$
- $\sigma$  = extinction coefficient (absorption + scattering)

The integral represents in-scattering from all directions

**Translation:** The change in radiance along a ray equals the loss due to extinction minus the gain from scattering from all other directions.

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#### Step 2: Define Configuration A

**Configuration A:** Source in direction  $\Omega'$ , detector in direction  $\Omega$

An external source illuminates the medium with radiance  $I_0$  along direction  $\Omega'$ . A detector at position  $\mathbf{r}_d$  measures scattered radiance in direction  $\Omega$ . The boundary condition at the source location  $\mathbf{r}_s$  is:

$$I(r_s, \Omega') = I_0$$

The detector measures:

$$I_A = I(r_d, \Omega)$$

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### Step 3: Define Configuration B

**Configuration B:** Source in direction  $\Omega$ , detector in direction  $\Omega'$

Now we swap: source illuminates along direction  $\Omega$  with the same radiance  $I_0$ , and the detector measures along direction  $\Omega'$ . The boundary condition is:

$$I(r_d, \Omega) = I_0$$

The detector measures:

$$I_B = I(r_s, \Omega')$$

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### Step 4: Key Property of the Phase Function

The phase function satisfies reciprocity:

$$P(\Omega' \rightarrow \Omega) = P(\Omega \rightarrow \Omega')$$

**Translation:** The probability of scattering from direction  $\Omega'$  to direction  $\Omega$  equals the probability of scattering from  $\Omega$  to  $\Omega'$ . This is a fundamental property of elastic scattering, derived from **time-reversal symmetry** in physics. This means the scattering process itself is symmetric with respect to direction reversal.

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### Step 5: Formal Solution Using Green Functions

The solution to the radiative transfer equation can be written using the **Green function**  $G(\mathbf{r}_d, \Omega; \mathbf{r}_s, \Omega')$ , which represents the **radiance** at  $(\mathbf{r}_d, \Omega)$  due to a **unit source** at  $(\mathbf{r}_s, \Omega')$ :

$$I(r_d, \Omega) = I_0 G(r_d, \Omega; r_s, \Omega')$$

The Green function satisfies:

$$\Omega \cdot \nabla G(r, \Omega; r_s, \Omega') = -\sigma G(r, \Omega; r_s, \Omega') + \sigma'_s \int_{4\pi} P(\Omega'' \rightarrow \Omega) G(r, \Omega''; r_s, \Omega') d\Omega''$$

with boundary condition:

$$G(r_s, \Omega'; r_s, \Omega') = \delta(0)$$

### Step 6: Reciprocity of the Green Function

The key theorem: The Green function is symmetric:

$$G(r_d, \Omega; r_s, \Omega') = G(r_s, \Omega'; r_d, \Omega)$$

**Proof sketch (See Appendix A):** This can be proven using the **adjoint operator** method. Consider two Green functions  $G_1$  and  $G_2$  corresponding to different source positions and directions. **Multiply** the RTE for  $G_1$  by  $G_2$  and vice versa, then **integrate** over the entire volume and all directions. The **extinction and scattering terms cancel** due to the reciprocity of the phase function, **leaving only boundary terms that prove the symmetry**.

### Step 7: Apply Reciprocity to Our Problem

For Configuration A:

$$I_A = I(r_d, \Omega) = I_0 G(r_d, \Omega; r_s, \Omega')$$

For Configuration B:

$$I_B = I(r_s, \Omega') = I_0 G(r_s, \Omega'; r_d, \Omega)$$

Using the reciprocity of the Green function:

$$G(r_d, \Omega; r_s, \Omega') = G(r_s, \Omega'; r_d, \Omega)$$

Therefore:

$$I_A = I_B$$

**This proves reciprocity!**

### Physical Interpretation

The reciprocity principle for scattering media works because:

1. **Phase function symmetry** - The scattering phase function ensures that the probability of scattering is the same in both directions

2. **Time-reversal symmetry** - For elastic scattering (no energy change), reversing the direction of propagation gives the same physical process

3. **Linear superposition** - The radiative transfer equation is linear, so the principle of superposition applies to all scattering paths

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### **Application: Vegetation Remote Sensing**

**Problem Setup:** Consider a vegetation canopy (forest, crop field, grassland) illuminated by sunlight. The canopy consists of leaves that both absorb and scatter radiation. We want to understand how satellite or airborne sensors detect the scattered light.

#### **Canopy Properties:**

- Leaves absorb radiation (chlorophyll absorption in visible, water absorption in infrared)
- Leaves scatter radiation (surface reflection, internal scattering)
- Extinction coefficient depends on leaf area density
- Phase function depends on leaf orientation distribution

#### **Configuration A: Solar Illumination with Satellite Observation**

- Source direction: Solar illumination (e.g., sun at 30 degrees from zenith)
- Detector direction: Satellite view (e.g., nadir looking down)
- Measured radiance by satellite

#### **Configuration B: Reversed Geometry**

- Source direction: Illumination from nadir (hypothetically)
- Detector direction: Detection at 30 degrees from zenith
- Measured radiance in this configuration

**Reciprocity Result:** The measured radiances are equal. This means that if we could illuminate the canopy from the satellite and observe from the solar direction, we would measure the same radiance as in the normal configuration. This is extremely useful for:

1. **Bidirectional Reflectance Distribution Function (BRDF) modeling** - We can simplify BRDF calculations by exploiting symmetry

2. **Multi-angle remote sensing** - Data from different view and solar geometries can be related through reciprocity

3. **Computational efficiency** - Only need to compute half of the angular configurations; the rest follow from reciprocity

**Practical Note:** In vegetation remote sensing, the phase function depends on leaf angle distribution. For randomly oriented leaves (spherical distribution), the phase function is nearly isotropic and clearly satisfies reciprocity. For oriented canopies (like grass or crops with preferred leaf angles), the phase function is anisotropic but still satisfies reciprocity.

**Conclusion:** The principle of reciprocity in scattering media states that exchanging the source and detector positions and directions yields the same detected radiance. This fundamental result stems from the symmetry of the scattering phase function and the linearity of the radiative transfer equation. In vegetation remote sensing, this principle simplifies BRDF modeling, enables efficient multi-angle observations, and provides a consistency check for radiative transfer models.

## Appendix A

### Detailed Mathematical Proof of Green Function Reciprocity

#### for Absorbing and Scattering Media

##### Introduction

We will prove that the Green function for the radiative transfer equation satisfies the reciprocity property. This proof uses the adjoint operator method and demonstrates that the symmetry of the phase function leads to symmetry of the Green function.

##### Step 1: Define the Green Function

The Green function  $G$  satisfies the radiative transfer equation:

$$\Omega \cdot \nabla G(r, \Omega; r_0, \Omega_0) = -\sigma G(r, \Omega; r_0, \Omega_0) + \sigma'_s \int_{4\pi} P(\Omega' \rightarrow \Omega) G(r, \Omega'; r_0, \Omega_0) d\Omega'$$

where  $G(\mathbf{r}, \boldsymbol{\Omega}; \mathbf{r}_0, \boldsymbol{\Omega}_0)$  represents the radiance at position  $\mathbf{r}$  in direction  $\boldsymbol{\Omega}$  due to a unit source at position  $\mathbf{r}_0$  in direction  $\boldsymbol{\Omega}_0$ .

##### Step 2: Consider Two Green Functions

Consider two Green functions with different source parameters:

$$G_1 = G(\mathbf{r}, \boldsymbol{\Omega}; \mathbf{r}_s, \boldsymbol{\Omega}_s) \text{ - source at } (\mathbf{r}_s, \boldsymbol{\Omega}_s)$$

$$G_2 = G(\mathbf{r}, \boldsymbol{\Omega}; \mathbf{r}_d, \boldsymbol{\Omega}_d) \text{ - source at } (\mathbf{r}_d, \boldsymbol{\Omega}_d)$$

##### Step 3: Write the RTE for Both Green Functions

For  $G_1$ :

$$\Omega \cdot \nabla G_1 = -\sigma G_1 + \sigma'_s \int_{4\pi} P(\Omega' \rightarrow \Omega) G_1(r, \Omega') d\Omega'$$

For  $G_2$ :

$$\Omega \cdot \nabla G_2 = -\sigma G_2 + \sigma'_s \int_{4\pi} P(\Omega' \rightarrow \Omega) G_2(r, \Omega') d\Omega'$$

##### Step 4: Multiply and Integrate

Multiply the first equation by  $G_2$  and the second by  $G_1$ , then integrate over all positions  $\mathbf{r}$  and all directions  $\boldsymbol{\Omega}$ :

$$\int_V \int_{4\pi} G_2 [\boldsymbol{\Omega} \cdot \nabla G_1 + \sigma G_1] d\Omega dr = \int_V \int_{4\pi} \sigma'_s G_2 \int_{4\pi} P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) G_1 d\Omega' d\Omega dr$$

Similarly:

$$\int_V \int_{4\pi} G_1 [\boldsymbol{\Omega} \cdot \nabla G_2 + \sigma G_2] d\Omega dr = \int_V \int_{4\pi} \sigma'_s G_1 \int_{4\pi} P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) G_2 d\Omega' d\Omega dr$$

### Step 5: Subtract the Two Equations

Subtracting the second equation from the first:

$$\int_V \int_{4\pi} [G_2 \boldsymbol{\Omega} \cdot \nabla G_1 - G_1 \boldsymbol{\Omega} \cdot \nabla G_2] d\Omega dr = \text{scattering terms}$$

Key observation: The extinction terms (with  $\sigma$ ) cancel on the left side because they appear with opposite signs in the two equations.

### Step 6: Apply Divergence Theorem to Left Side

The left side can be rewritten using the divergence theorem. Note that:

$$\boldsymbol{\Omega} \cdot \nabla (G_2 G_1) = G_2 \boldsymbol{\Omega} \cdot \nabla G_1 + G_1 \boldsymbol{\Omega} \cdot \nabla G_2$$

Therefore:

$$G_2 \boldsymbol{\Omega} \cdot \nabla G_1 - G_1 \boldsymbol{\Omega} \cdot \nabla G_2 = \nabla \cdot (\boldsymbol{\Omega} G_2 G_1)$$

Applying the divergence theorem converts the volume integral to a surface integral:

$$\int_V \int_{4\pi} \nabla \cdot (\boldsymbol{\Omega} G_2 G_1) d\Omega dr = \int_S \int_{4\pi} \boldsymbol{\Omega} \cdot \mathbf{n} G_2 G_1 d\Omega dS$$

where  $\mathbf{n}$  is the outward normal to the surface S.

### Step 7: Evaluate the Scattering Terms Using Phase Function Reciprocity

The right-hand side contains the scattering integrals. Using the phase function reciprocity:

$$P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) = P(\boldsymbol{\Omega} \rightarrow \boldsymbol{\Omega}')$$

Consider the scattering term from the first equation:

$$\int_{4\pi} \int_{4\pi} \sigma'_s P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) G_2(\boldsymbol{\Omega}) G_1(\boldsymbol{\Omega}') d\Omega' d\Omega$$

Now swap the dummy variables  $\boldsymbol{\Omega}$  and  $\boldsymbol{\Omega}'$  in this integral:

$$\int_{4\pi} \int_{4\pi} \sigma'_s P(\Omega \rightarrow \Omega') G_2(\Omega') G_1(\Omega) d\Omega d\Omega'$$

Using  $P(\Omega \rightarrow \Omega') = P(\Omega' \rightarrow \Omega)$ , we see that this is exactly equal to the scattering term from the second equation. Therefore, the scattering terms on the right-hand side are identical and cancel when we subtract!

### Step 8: Apply Boundary Conditions

We are left with:

$$\int_S \int_{4\pi} \Omega \cdot n G_2 G_1 d\Omega dS = 0$$

For a medium with vacuum boundaries (no incoming radiation from outside), and if we take the surface S to infinity where the Green functions vanish, this boundary integral equals zero.

This proves that the equation holds for any choice of source positions and directions.

### Step 9: Conclusion - Green Function Reciprocity

From the analysis above, we have shown that the integral equation is satisfied when:

$$G(r_d, \Omega; r_s, \Omega') = G(r_s, \Omega'; r_d, \Omega)$$

**This is the reciprocity property of the Green function.**

### Summary

The key steps in the proof were:

1. Write the RTE for two different Green functions with different source parameters
2. Multiply each equation by the other Green function and integrate over all space and directions
3. Subtract the two resulting equations
4. Observe that extinction terms cancel due to symmetry
5. Apply the reciprocity of the phase function to show scattering terms are equal
6. Use the divergence theorem to convert volume integrals to surface integrals
7. Apply appropriate boundary conditions at infinity
8. Conclude that the Green function must be symmetric

The crucial physical ingredient is the reciprocity of the phase function:  $P(\boldsymbol{\Omega}' \rightarrow \boldsymbol{\Omega}) = P(\boldsymbol{\Omega} \rightarrow \boldsymbol{\Omega}')$ , which follows from time-reversal symmetry in elastic scattering.