EE 445/645: Physical Models in Remote Sensing (Spring 2026)

Chapter 01-Part 06: Energy Balance – Notes (See C1-P6 PPTs)

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1. Introduction

In this document, we will integrate the radiative transfer equation (RTE) over a three-dimensional spatial domain *V* and over all solid angles on a sphere (*4π* steradians). This integration is fundamental in radiation transport theory and leads to important conservation laws. The starting equation is:

$$\mathbf{\Omega} \cdot \nabla I + \sigma(\mathbf{r},\mathbf{\Omega})\, I(\mathbf{r},\mathbf{\Omega}) = \int_{4\pi} d\Omega'\, \sigma_s(\mathbf{r},\Omega' \rightarrow \mathbf{\Omega})\, I(\mathbf{r},\Omega') + Q(\mathbf{r},\mathbf{\Omega})$$

where (note: vectors are denoted in **boldface**):

- $I(\mathbf{r},\mathbf{\Omega})$ is the radiation intensity at position $\mathbf{r}$ propagating in direction $\mathbf{\Omega}$
- $\mathbf{\Omega}$ is the unit direction vector (points in the direction of photon travel)
- ∇ is the gradient operator with respect to position $\mathbf{r}$ (spatial derivative)
- $\sigma(\mathbf{r},\mathbf{\Omega})$ is the total extinction coefficient (units: m^{-1})
- $\sigma_s(\mathbf{r},\Omega' \rightarrow \mathbf{\Omega})$ is the differential scattering coefficient (units: $m^{-1}\ sr^{-1}$) describing scattering from direction Ω' into direction $\mathbf{\Omega}$
- $\sigma'_s(\mathbf{r})$ is the total scattering coefficient (units: m^{-1}) — note that $\sigma'_s = \int_{4\pi} d\Omega\, \sigma_s(\Omega' \rightarrow \mathbf{\Omega})$
- $Q(\mathbf{r},\mathbf{\Omega})$ is the volumetric source term (emission)

The notation $\int_{4\pi}$ means integration over the full sphere (*4π* steradians).

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2. Step 1: Integrate Over All Directions

We integrate both sides over all solid angles $\int_{4\pi} d\Omega$:

$$\int_{4\pi} d\Omega \, [\mathbf{\Omega} \cdot \nabla I \; + \; \sigma(\mathbf{r},\mathbf{\Omega}) \, I(\mathbf{r},\mathbf{\Omega})] \; = \int_{4\pi} d\Omega \, [\int_{4\pi} d\Omega' \, \sigma_s(\mathbf{r},\Omega' \to \mathbf{\Omega}) \, I(\mathbf{r},\Omega') + \; Q(\mathbf{r},\mathbf{\Omega})]$$

2.1 Left-Hand Side: First Term (Streaming Term)

Consider:

$$\int_{4\pi} d\Omega \, (\mathbf{\Omega} \cdot \nabla I)$$

The gradient ∇ acts on spatial coordinates $\boldsymbol{r}$, while $\int_{4\pi} d\Omega$ integrates over the angular variable $\boldsymbol{\Omega}$. Since they operate on different independent variables, they commute:

$$\int_{4\pi} d\Omega \, (\mathbf{\Omega} \cdot \nabla I) \; = \; \nabla \cdot \int_{4\pi} d\Omega \, (\mathbf{\Omega} \, I)$$

We define the radiation flux vector:

$$\mathbf{F}(\mathbf{r}) \; = \; \int_{4\pi} d\Omega \, \mathbf{\Omega} \, I(\mathbf{r},\mathbf{\Omega})$$

Thus:

$$\int_{4\pi} d\Omega \, (\mathbf{\Omega} \cdot \nabla I) \; = \; \nabla \cdot \mathbf{F}(\mathbf{r}) = \text{Flux Net Outflow}$$

2.2 Left-Hand Side: Second Term (Extinction)

The extinction term integrates as:

$$\int_{4\pi} d\Omega \, \sigma(\mathbf{r},\mathbf{\Omega}) \, I(\mathbf{r},\mathbf{\Omega}) = \text{Extinction Flux}$$

2.3 Right-Hand Side: Scattering Integral

For the scattering term:

$$\int_{4\pi} d\Omega \, [\int_{4\pi} d\Omega' \, \sigma_s(\mathbf{r},\Omega' \to \mathbf{\Omega}) \, I(\mathbf{r},\Omega') \,] = \; \int_{4\pi} d\Omega \int_{4\pi} d\Omega' \, \sigma_s(\mathbf{r},\Omega' \to \mathbf{\Omega}) \, I(\mathbf{r},\Omega')$$

By conservation of scattering, the total scattering out of direction Ω' into all directions equals the total scattering coefficient $\sigma'_s(\boldsymbol{r})$:

$$\int_{4\pi} d\Omega \, \sigma_s(\mathbf{r},\Omega' \to \mathbf{\Omega}) \; = \; \sigma'_s(\mathbf{r},\Omega')$$

This is the definition of the total scattering coefficient (with prime) as distinct from the differential scattering coefficient (without prime). Applying this:

$$\int_{4\pi} d\Omega' \, \sigma'_s(\mathbf{r},\Omega') \, I(\mathbf{r},\Omega') \; = \; \text{Inscattered Flux}$$

2.4 Right-Hand Side: Source Term

$$\int_{4\pi} d\Omega\, Q(\mathbf{r},\mathbf{\Omega}) = \text{Source Flux}$$

2.5 Result After Angular Integration

Combining all terms:

$$\nabla \cdot \mathbf{F}(\mathbf{r}) + \int_{4\pi} d\Omega\, \sigma(\mathbf{r},\mathbf{\Omega})\, I(\mathbf{r},\mathbf{\Omega}) = \int_{4\pi} d\Omega'\, \sigma'_s(\mathbf{r},\Omega')\, I(\mathbf{r},\Omega') + \int_{4\pi} d\Omega\, Q(\mathbf{r},\mathbf{\Omega}) .$$

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3. Step 2: Integrate Over the Spatial Domain V

Now integrate over volume V. The volume element is $d^3\boldsymbol{r}$ and the integral is $\int_V d^3\boldsymbol{r}$ (where V appears as a subscript of equal size to $\int$):

$$\int_V d^3\mathbf{r}\, [\nabla \cdot \mathbf{F}(\mathbf{r}) + \int_{4\pi} d\Omega\, \sigma(\mathbf{r},\mathbf{\Omega})\, I(\mathbf{r},\mathbf{\Omega})] = \int_V d^3\mathbf{r}\, \int_{4\pi} d\Omega'\, \sigma'_s(\mathbf{r},\Omega')\, I(\mathbf{r},\Omega') + \int_{4\pi} d\Omega\, Q(\mathbf{r},\mathbf{\Omega})] .$$

3.1 Application of the Divergence Theorem

The divergence theorem (Gauss's theorem) relates a volume integral of a divergence to a surface integral over the boundary:

$$\int_V d^3\mathbf{r}\, (\nabla \cdot \mathbf{F}) = \int_{\partial V} dA\, (\mathbf{n} \cdot \mathbf{F}) ,$$

where:

- $\nabla \cdot \mathbf{F} = \partial F_x/\partial x + \partial F_\gamma/\partial y + \partial F_u/\partial z$ is the divergence in Cartesian coordinates
- V is the three-dimensional volume region
- ∂V is the closed surface boundary of V
- dA is a surface area element on ∂V
- $\mathbf{n}$ is the outward-pointing unit normal vector perpendicular to the surface
- $\mathbf{n} \cdot \mathbf{F}$ is the component of $\mathbf{F}$ perpendicular to the surface (flux through surface)

Physical meaning: The divergence $\nabla \cdot \boldsymbol{F}$ measures net outward flow per unit volume. If $\nabla \cdot \boldsymbol{F} > 0$, there is a net source (flow emanating outward); if $\nabla \cdot \boldsymbol{F} < 0$, there is a net sink (flow converging inward).

The surface integral $\int_{\partial V} dA\, (\boldsymbol{n} \cdot \boldsymbol{F})$ computes the total flux of $\mathbf{F}$ passing through the boundary surface ∂V in the outward direction. The dot product $\boldsymbol{n} \cdot \boldsymbol{F}$ picks out only the component of $\boldsymbol{F}$ that is perpendicular to (normal to) the surface.

The divergence theorem states that the total divergence (source/sink strength) within a volume equals the net flux leaving through its boundary.

Application to our problem: In our radiative transfer equation, the vector field is the radiation flux $\boldsymbol{F}(\boldsymbol{r})$. The quantity $\boldsymbol{n} \cdot \boldsymbol{F}$ represents the radiation flux component leaving the volume V through the surface element dA. Integrating over the entire boundary surface ∂V gives the total net radiation flux escaping the volume.

Therefore:

$$\int_V d^3\mathbf{r}\, (\nabla \cdot \mathbf{F}) = \int_{\partial V} dA\, (\mathbf{n} \cdot \mathbf{F}) = \text{Net radiation flux leaving } V\,.$$

3.2 Remaining Volume Integrals

Total extinction:

$$\int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, \sigma(\mathbf{r},\boldsymbol{\Omega})\, I(\mathbf{r},\boldsymbol{\Omega}) = \text{Total Extinction Flux}\,.$$

Total in-scattering:

$$\int_V d^3\mathbf{r} \int_{4\pi} d\Omega'\, \sigma'_s(\mathbf{r},\Omega')\, I(\mathbf{r},\Omega') = \text{Total Inscattering Flux}\,.$$

Total source:

$$\int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, Q(\mathbf{r},\boldsymbol{\Omega}) = \text{Total Source Flux}\,.$$

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4. Final Integrated Equation and Interpretation

4.1 The Doubly-Integrated Result

Combining all integrated terms:

$$\int_{\partial V} dA\, (\mathbf{n} \cdot \mathbf{F}) + \int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, \sigma(\mathbf{r},\boldsymbol{\Omega})\, I(\mathbf{r},\boldsymbol{\Omega}) =$$

$$\int_V d^3\mathbf{r} \int_{4\pi} d\Omega'\, \sigma'_s(\mathbf{r},\Omega')\, I(\mathbf{r},\Omega') + \int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, Q(\mathbf{r},\boldsymbol{\Omega})$$

results in

$$\text{Net Flux Leaving } V + \text{Total Extinction Flux} = \text{Total Inscattered Flux} + \text{Total Source Flux}\,.$$

Remember that,

$$\text{Total Extinction Flux} = \text{Total Absorption Flux} + \text{Total Outscattered Flux}\,.$$

The Total Outscattered Flux is equal to the Total Inscattered Flux at the domain scale because scattering results in neither gain nor loss of energy (it simply changes the direction of photon travel). Thus,

$$\text{Total Outscattered Flux} = \text{Total Inscattered Flux}\,.$$

Therefore,

Total Source Flux = Net Flux Leaving V + Total Absorbed .

4.2 Final Energy Balance Statement

Rearranging:

$$\int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, Q(\mathbf{r},\mathbf{\Omega}) = \int_{\partial V} dA\, (\mathbf{n} \cdot \mathbf{F}) + \int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, \sigma_a(\mathbf{r},\mathbf{\Omega})\, I(\mathbf{r},\mathbf{\Omega}) ,$$

Total Source = Net Outflow + Total Absorption .

This is the fundamental conservation law: All radiation energy generated within volume V (sources) must either escape through the boundary (net outflow) or be absorbed within the volume. Energy is conserved.

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5. Physical Interpretation

Total Source: $\int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, Q(\mathbf{r},\mathbf{\Omega})$

Total radiation emitted within V over all positions and directions. Includes thermal emission, fluorescence, external sources, etc.

Net Outflow: $\int_{\partial V} dA\, (\mathbf{n} \cdot \mathbf{F})$

Net radiation flux escaping through boundary ∂V. If $\boldsymbol{n} \cdot \boldsymbol{F} > 0$ at a point, radiation leaves; if negative, radiation enters.

Total Absorption: $\int_V d^3\mathbf{r} \int_{4\pi} d\Omega\, \sigma_a(\mathbf{r},\mathbf{\Omega})\, I(\mathbf{r},\mathbf{\Omega})$

Total radiation absorbed within V, converted to other energy forms (e.g., heat). Unlike scattering, absorption removes photons from the radiation field entirely.

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6. Applications

- **Astrophysics:** Energy budgets in stellar atmospheres, radiation pressure in star formation
- **Climate science:** Global energy balance, radiative forcing calculations
- **Nuclear engineering:** Neutron balance in reactor cores, criticality analysis
- **Medical physics:** Radiation dose planning, energy deposition in tissue
- **Computational validation:** If total source $\neq$ net outflow + absorption, the numerical code has an error

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7. Summary

1. **Angular integration:** Integrated over 4π steradians ($\int_{4\pi} d\Omega$), converting $\boldsymbol{\Omega} \cdot \nabla I$ *to* $\nabla \cdot \boldsymbol{F}$, and introducing σ'_s (total scattering) from σ_s (differential scattering)

2. **Spatial integration:** Integrated over V ($\int_V d^3\mathbf{r}$), applying divergence theorem to convert $\nabla \cdot \boldsymbol{F}$ into surface flux $\int_{\partial V} dA\ (\boldsymbol{n} \cdot \boldsymbol{F})$

3. **Absorption:** Used $\sigma = \sigma_a + \sigma'_s$ to show extinction minus in-scattering equals absorption

4. **Result:** Total source = net outflow + total absorption — the global energy conservation law for radiation