EE 445/645: Physical Models in Remote Sensing (Spring 2026)

Chapter 01-Part 05: The Radiative Transfer Equation – Notes (See C1-P5 PPTs)

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1. Introduction to Time-Independent Radiative Transfer

1.1 Stationary Conditions

In this chapter, we focus on the time-independent (stationary) form of the Radiative Transfer Equation (RTE). This simplification is valid under several important assumptions that frequently apply to remote sensing scenarios.

No internal sources: There are no emitting sources within the host medium itself. All radiation originates from boundary conditions (e.g., solar radiation entering from above, surface emission from below).

Stationary cross sections: The absorption and scattering cross sections of the medium do not change with time. The optical properties remain constant during observation.

No frequency shifting: Scattering interactions do not change the frequency or wavelength of photons. This excludes inelastic scattering like Raman scattering but includes elastic scattering like Rayleigh and Mie scattering.

Non-re-entrant medium: Once photons leave the medium, they cannot re-enter. This applies to plane-parallel geometries.

1.2 Physical Significance

These assumptions allow us to develop an integral formulation of the RTE that is particularly useful for remote sensing. The stationary assumption is excellent for:

- Satellite observations where measurement time scales (milliseconds) are much shorter than atmospheric changes (minutes to hours)
- Analysis of reflected solar radiation in visible and near-infrared spectrum
- Thermal infrared emission under quasi-steady state conditions

2. Radiative Transfer in Vacuum

2.1 The Simplest Case

The vacuum problem represents the absolute simplest radiative transfer scenario. In vacuum, there is no medium present - no absorption, no scattering, no emission. Photons simply travel in straight lines at the speed of light.

The RTE in vacuum reduces to:

$$\boldsymbol{\Omega} \cdot \nabla I = 0$$

where $\boldsymbol{\Omega}$ is the direction vector and I is the radiance (intensity). This equation states that the directional derivative of intensity along the direction of propagation is zero.

This means that radiance is constant along any ray path. A photon traveling through vacuum maintains its energy and direction indefinitely. The solution is:

$$\mathrm{I}(\mathbf{r}, \boldsymbol{\Omega}) = \mathrm{I}(\mathbf{r}_{\mathrm{B}}, \boldsymbol{\Omega})$$

where $\mathbf{r}_B$ is any point along the ray and $\mathbf{r}$ is the observation point.

2.2 Applications in Remote Sensing

Space-based observations: The vacuum solution applies to radiation propagating through space from a satellite to the top of Earth's atmosphere. This is why satellites can observe Earth from hundreds of kilometers away - the radiation traveled through vacuum without alteration.

Inverse square law: While intensity along a ray is constant, the irradiance (flux) at a surface decreases as $1/r^2$ due to geometric spreading. This is why solar irradiance at Earth is much less than at the Sun's surface, even though individual photons don't lose energy.

3. Atmospheric Refraction and Stellar Twinkling

3.1 Refractive Index Fundamentals

The refractive index n describes how electromagnetic radiation propagates through a material compared to vacuum. It is defined as n = c/v, where c is the speed of light in vacuum (approximately 3×10^8 m/s) and v is the speed of light in the medium.

- Vacuum has n = 1 by definition
- Air at sea level has n ≈ 1.0003 (very close to vacuum)
- Water has n ≈ 1.33
- Glass has n ≈ 1.5 to 1.9 depending on composition

3.2 Snell's Law and Refraction

When light passes from one medium to another with different refractive indices, it bends according to Snell's law:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Physical Cause: Refraction occurs because light travels at different speeds in different media. When a wavefront enters a medium at an angle, one side slows down before the other, causing the wave to bend - similar to how a car turns when one wheel hits sand.

3.3 Why Stars Twinkle

Stars appear to twinkle (scintillate) due to atmospheric turbulence. This phenomenon does NOT occur in space - astronauts see stars as steady points of light.

Mechanism:

1. Earth's atmosphere contains pockets of air with different temperatures and densities

2. Temperature variations cause refractive index variations (warmer air has lower density and refractive index)

3. As starlight passes through turbulent air, it gets refracted in different directions

4. Turbulent cells move and change on timescales of milliseconds to seconds

5. The result is that the star's apparent position and brightness fluctuate - we see it twinkle

Why don't planets twinkle as much?

Planets appear as small disks rather than point sources. Different parts of the disk are affected by different turbulent cells, and the fluctuations average out. This is why Venus, Jupiter, and Mars appear as steady lights.

4. Radiative Transfer in Absorbing and Non-Scattering Media

4.1 The Uncollided Radiation Field

When we add absorption but exclude scattering, we get a fundamental result. The radiation field consists only of photons that have NOT interacted with the medium - the "uncollided" photons.

The RTE becomes:

$$\mathbf{\Omega} \cdot \nabla I_0 + \sigma_a I_0 = 0$$

where σ_a is the absorption cross section and I_0 denotes the uncollided intensity. The subscript "0" emphasizes that these photons have experienced zero collisions.

4.2 Solution: The Beer-Lambert Law

The solution is the famous Beer-Lambert law (also called Beer's law or the Beer-Lambert-Bouguer law):

$$I_0(\mathbf{r}, \boldsymbol{\Omega}) = I_0(\mathbf{r}_B, \boldsymbol{\Omega})\exp(-\tau)$$

where τ is the optical depth from boundary point $\mathbf{r}_B$ to observation point $\mathbf{r}$.

The derivation of the integral equation of transfer is given in **Appendix A** at the end of these notes.

Optical Depth Definition: The optical depth is defined as the integral of the extinction coefficient along the path:

$$\tau = \int_{\mathbf{r}_B}^{\mathbf{r}} \sigma_a(\mathbf{r}', \boldsymbol{\Omega})ds'$$

4.3 Physical Interpretation

Exponential Decay: The Beer-Lambert law describes exponential attenuation. Each infinitesimal layer removes a fixed fraction of radiation passing through it. This leads to exponential - not linear - decay.

Survival Probability: The factor $\exp(-\tau)$ is the probability that a photon survives (is not absorbed) while traveling from $\mathbf{r}_B$ to $\mathbf{r}$. If $\tau = 1$, survival probability is $e^{-1} \approx 0.37$ or 37%. If $\tau = 2$, only $e^{-2} \approx 0.14$ or 14% survive.

4.4 Optical Depth as a Natural Scale

Optical depth provides a natural dimensionless measure of how "thick" a medium is to radiation:

τ << 1: Optically thin - most photons pass through unabsorbed ($e^{-0.1}$ = 90% transmission)

$\tau \approx$ 1: Moderate optical depth - significant absorption (37% transmission)

τ >> 1: Optically thick - very little radiation penetrates (e^{-5} = 0.7% transmission)

5. The Gap Fraction and Applications

5.1 Definition and Concept

The gap fraction (also called gap probability or canopy gap fraction) is defined as the probability that a ray can pass through a medium without being intercepted. It equals $\exp(-\tau)$, which is exactly the transmittance from Beer-Lambert law.

Notation: Gap fraction is often denoted as $P_0(\theta)$, where θ is the zenith angle. The subscript "0" indicates zero collisions - photons that made it through without hitting anything.

5.2 Applications in Vegetation Remote Sensing

Leaf Area Index (LAI) Estimation: For vegetation canopies, the gap fraction can be measured using hemispherical photography or LiDAR. The optical depth is related to LAI through:

$$\tau(\theta) = \frac{G(\theta) \times LAI}{\cos(\theta)}$$

where G(θ) is the projection function describing leaf angle distribution.

Forest Structure: Measuring gap fraction at multiple angles provides information about canopy architecture. Dense canopies have low gap fractions (high τ), while open canopies have high gap fractions (low τ). Angular variation reveals information about leaf clumping and tree spacing.

6. Weakly Scattering Media: Single Scattering Approximation

6.1 Introduction to Scattering

When we add scattering, the mathematics becomes significantly more complex. However, if scattering is weak, we can use a perturbative approach: calculate the effect of one scattering event and stop.

When is single scattering valid?

- When the scattering optical depth is small ($\tau_s \ll 1$)
- When the medium is primarily absorbing with weak scattering (single scattering albedo $\omega_0 \ll 1$)
- For thin layers or dilute media where most photons either pass through unscattered or scatter once at most

6.2 The Once-collided Radiation Field

The RTE becomes:

$$\underline{\Omega} \cdot \nabla I_1 + \sigma(\vec{\mathbf{r}},\underline{\boldsymbol{\Omega}})\, I_1(\vec{\mathbf{r}},\underline{\boldsymbol{\Omega}}) = \int_{4\pi} d\underline{\boldsymbol{\Omega}}'\, \sigma_s(\vec{\mathbf{r}},\underline{\boldsymbol{\Omega}}' \rightarrow \underline{\boldsymbol{\Omega}})\, I_0(\vec{\mathbf{r}},\underline{\boldsymbol{\Omega}}')\,.$$

Remember the term on the RHS of the above RTE is the first collision source (source of once-collided photons). It is known because the uncollided intensity I_0 is known (see Section 4.2). The solution to the above is,

$$I^1(\vec{\mathbf{r}},\underline{\boldsymbol{\Omega}}) = \int_0^{|\vec{\mathbf{r}}-\vec{\mathbf{r}}\mathbf{B}|} ds'\, Q^1\left(\vec{\mathbf{r}} - s'\underline{\boldsymbol{\Omega}}, \underline{\boldsymbol{\Omega}}\right) \exp\left[-\int_0^{s'} ds''\sigma(\vec{\mathbf{r}} - s''\underline{\boldsymbol{\Omega}}, \underline{\boldsymbol{\Omega}})\right]\,.$$

The derivation is given **Appendix B** at the end of these notes.

7. Multiple Scattering and Successive Orders

7.1 The Method of Successive Orders of Scattering

When scattering is not weak, photons can scatter multiple times. We organize the solution by counting scattering events:

I_0: Uncollided intensity (zero scattering events)

I_1: Once-scattered intensity (one scattering event)

I_2: Twice-scattered intensity (two scattering events)

I_N: N-times-scattered intensity

The total intensity is the sum of all orders:

$$\mathrm{I} = \mathrm{I}_0 + \mathrm{I}_1 + \mathrm{I}_2 + \ldots + \mathrm{I}_\mathrm{N}$$

For most physical problems, this series converges. Each successive order contributes less because each scattering event attenuates radiation and dilutes intensity by scattering into 4π directions.

8. The Integral Equation of Transfer

8.1 General Formulation

All previous cases can be unified into a single elegant formulation - the integral equation of transfer. This equation expresses intensity at any point as:

1. Attenuated boundary radiation

2. Attenuated contributions from sources along the ray

$$\mathrm{I}(\mathbf{r},\boldsymbol{\Omega}) = \mathrm{I}(\mathbf{r}_\mathrm{B},\boldsymbol{\Omega})\exp\left(-\tau(\mathbf{r},\mathbf{r}_\mathrm{B})\right) + \int_0^{s_\mathrm{B}} \mathrm{S}(\mathbf{r}(s'),\boldsymbol{\Omega})\exp\left(-\tau\left(\mathbf{r},\mathbf{r}(s')\right)\right)\mathrm{d}s'$$

The source function S(**r**,**Ω**) contains all the physics of how radiation is generated or redistributed at each point:

$$\mathrm{S}(\mathbf{r},\boldsymbol{\Omega}) = \left[\iint_{4\pi} \sigma_s(\mathbf{r},\boldsymbol{\Omega}' \to \boldsymbol{\Omega})\mathrm{I}(\mathbf{r},\boldsymbol{\Omega}')\mathrm{d}\boldsymbol{\Omega}'\right]$$

The derivation of the integral equation of transfer is given in **Appendix C** at the end of these notes.

9. Historical Perspective: Subrahmanyan Chandrasekhar

Subrahmanyan Chandrasekhar (1910-1995) was one of the 20th century's greatest astrophysicists. He made fundamental contributions to stellar structure, radiative transfer, black holes, and many other areas.

Key Achievements:

Nobel Prize in Physics (1983): For theoretical studies of physical processes important to stellar structure and evolution

Chandrasekhar Limit: Maximum mass (~1.4 solar masses) of a white dwarf before it collapses

Radiative Transfer (1950): Definitive mathematical treatment still widely cited today

Appendix A

Solution to the Radiative Transfer Equation for Uncollided Intensity

Problem Statement

Given the radiative transfer equation:

$dI(\mathbf{r} + s\mathbf{\Omega}, \mathbf{\Omega})/ds + \sigma_\lambda(\mathbf{r}, \mathbf{\Omega})\, I_\lambda(\mathbf{r}, \mathbf{\Omega}) = 0$

With boundary condition:

$I(\mathbf{r}, \mathbf{\Omega}) = B(\mathbf{r}_B, \mathbf{\Omega})$

where $\mathbf{r}_B \in \delta V$, $[\mathbf{n}_B(\mathbf{r}_B) \cdot \mathbf{\Omega}] < 0$

Find: $I(\mathbf{r}, \mathbf{\Omega})$

Step 1: Understanding the Equation

This is a first-order linear ordinary differential equation (ODE) along a ray path parameterized by s. The equation can be written as:

$dI/ds + \sigma_\lambda I_\lambda = 0$

where:

- $I = I_\lambda(\mathbf{r} + s\mathbf{\Omega}, \mathbf{\Omega})$ is the intensity at position $\mathbf{r} + s\mathbf{\Omega}$ in direction $\mathbf{\Omega}$
- $\sigma_\lambda = \sigma_\lambda(\mathbf{r}, \mathbf{\Omega})$ is the extinction coefficient
- s is the arc length parameter along the ray

Step 2: Solve the Differential Equation

Starting with:

$dI/ds = -\sigma_\lambda I$

Separate variables:

$dI/I = -\sigma_\lambda\, ds$

Integrate both sides:

$\int dI/I = -\int \sigma_\lambda\, ds$

This gives:

$\ln(I) = -\int_0^s \sigma_\lambda(\mathbf{r} + s'\mathbf{\Omega}, \mathbf{\Omega})\, ds' + C$

where C is the constant of integration.

Exponentiate both sides:

$I(s) = I_0 \exp(-\int_0^s \sigma_\lambda(\mathbf{r} + s'\mathbf{\Omega}, \mathbf{\Omega})\, ds')$

where $I_0 = e^C$ is determined by the boundary condition.

Step 3: Define the Optical Depth

The optical depth (or optical thickness) τ is defined as:

$$\tau(s) = \int_0^s \sigma_\lambda(\mathbf{r} + s'\mathbf{\Omega}, \mathbf{\Omega})\, ds'$$

This represents the cumulative extinction along the ray path from $s = 0$ to s. The solution becomes:

$$I(s) = I_0 \exp(-\tau(s))$$

Step 4: Apply the Boundary Condition

The boundary condition states that at the boundary point $\mathbf{r}_B$ (where the ray enters the volume):

$$I(\mathbf{r}_B, \mathbf{\Omega}) = B(\mathbf{r}_B, \mathbf{\Omega})$$

The condition $[\mathbf{n}_B(\mathbf{r}_B) \cdot \mathbf{\Omega}] < 0$ ensures the ray is entering (not exiting) the volume. Let $s = 0$ correspond to the boundary point $\mathbf{r}_B$, so:

- $\mathbf{r} = \mathbf{r}_B + s\mathbf{\Omega}$
- At $s = 0$: $\mathbf{r} = \mathbf{r}_B$
- $I(0) = B(\mathbf{r}_B, \mathbf{\Omega})$

Therefore, $I_0 = B(\mathbf{r}_B, \mathbf{\Omega})$.

Step 5: General Solution

The complete solution is:

$$I(\mathbf{r}, \mathbf{\Omega}) = B(\mathbf{r}_B, \mathbf{\Omega}) \exp(-\tau(\mathbf{r}_B \to \mathbf{r}))$$

where:

$$\tau(\mathbf{r}_B \to \mathbf{r}) = \int_0^s \sigma_\lambda(\mathbf{r}_B + s'\mathbf{\Omega}, \mathbf{\Omega})\, ds'$$

and s is chosen such that $\mathbf{r} = \mathbf{r}_B + s\mathbf{\Omega}$. Alternatively, we can write:

$$\tau(\mathbf{r}_B \to \mathbf{r}) = \int_{\mathbf{r}_B}^{\mathbf{r}} \sigma_\lambda(\mathbf{r}', \mathbf{\Omega})\, |d\mathbf{r}'|$$

where the integral is along the ray path from $\mathbf{r}_B$ to $\mathbf{r}$.

Physical Interpretation

The intensity **decreases exponentially** along the ray path due to absorption/extinction:

$$I(\mathbf{r}, \mathbf{\Omega}) = B(\mathbf{r}_B, \mathbf{\Omega}) \times \exp(-\tau)$$

- If $\tau \ll 1$ (optically thin): $I \approx B$ (little extinction)
- If $\tau \gg 1$ (optically thick): $I \approx 0$ (strong extinction)
- The factor $\exp(-\tau)$ is the **transmittance** of the mediu

Appendix B

Solution to the Radiative Transfer Equation for Once-Collided Intensity

Problem Statement

Solve the radiative transfer equation (RTE) with scattering:

$$\mathbf{\Omega} \cdot \nabla I^1 + \sigma(\mathbf{r}, \mathbf{\Omega})\, I^1(\mathbf{r}, \mathbf{\Omega}) = \int d\mathbf{\Omega}'/4\pi\, \sigma_s(\mathbf{r}, \mathbf{\Omega}' \to \mathbf{\Omega})\, I^0(\mathbf{r}, \mathbf{\Omega}') \quad (1)$$

with boundary condition:

$$I^1(\mathbf{r}, \mathbf{\Omega}) = 0, \quad \mathbf{r}_B \in \delta V, \quad [\mathbf{n}_B(\mathbf{r}_B) \cdot \mathbf{\Omega}] < 0 \quad (2)$$

Notation:

- $I^1(\mathbf{r}, \mathbf{\Omega})$ = once-collided intensity (what we seek)
- $I^0(\mathbf{r}, \mathbf{\Omega})$ = uncollided intensity (known from previous analysis)
- $\sigma(\mathbf{r}, \mathbf{\Omega})$ = total extinction coefficient
- $\sigma_s(\mathbf{r}, \mathbf{\Omega}' \to \mathbf{\Omega})$ = differential scattering cross-section
- $\mathbf{r}_B$ = boundary point where ray enters the domain
- $\mathbf{n}_B$ = outward normal at boundary
- The condition $[\mathbf{n}_B \cdot \mathbf{\Omega}] < 0$ ensures $\mathbf{\Omega}$ points inward

Physical Interpretation

The RHS of equation (1) represents the **first collision source**:

$$Q^1(\mathbf{r}, \mathbf{\Omega}) = \int d\mathbf{\Omega}'/4\pi\, \sigma_s(\mathbf{r}, \mathbf{\Omega}' \to \mathbf{\Omega})\, I^0(\mathbf{r}, \mathbf{\Omega}') \quad (3)$$

This represents photons that:

1. Traveled from an external source as uncollided radiation I^0
2. Scattered exactly once at position $\mathbf{r}$ from direction $\mathbf{\Omega}'$ into direction $\mathbf{\Omega}$
3. Now contribute to the once-collided intensity

Solution Method: Method of Characteristics

Step 1: Transform to Ray Coordinates

Consider a ray in direction $\mathbf{\Omega}$ passing through point $\mathbf{r}$. Parameterize positions along this ray:

$$\mathbf{r}(s) = \mathbf{r} - s\mathbf{\Omega} \quad (4)$$

where $s \geq 0$ measures distance backward along the ray from $\mathbf{r}$.

At $s = 0$: position is $\mathbf{r}$ (observation point)
At $s = S$: position is $\mathbf{r}_B$ (boundary point)

Therefore: $S = |\mathbf{r} - \mathbf{r}_B|$ along the ray direction.

Step 2: Convert to ODE Along Ray

The directional derivative operator becomes:

$\mathbf{\Omega} \cdot \nabla = -d/ds$ (5)

Substituting into equation (1):

$-dI^1/ds + \sigma(\mathbf{r}(s), \mathbf{\Omega})\, I^1(\mathbf{r}(s), \mathbf{\Omega}) = Q^1(\mathbf{r}(s), \mathbf{\Omega})$ (6)

Rearranging:

$dI^1/ds - \sigma I^1 = -Q^1$ (7)

This is a first-order linear ODE along the ray path.

Step 3: Solve Using Integrating Factor

The integrating factor for equation (7) is:

$\mu(s) = \exp[-\int_0^s \sigma(\mathbf{r}(s''), \mathbf{\Omega})\, ds'']$ (8)

Define the optical depth from observation point to distance s:

$\tau(s) = \int_0^s \sigma(\mathbf{r}(s''), \mathbf{\Omega})\, ds'' = \int_0^s \sigma(\mathbf{r} - s''\mathbf{\Omega}, \mathbf{\Omega})\, ds''$ (9)

Then $\mu(s) = \exp[-\tau(s)]$.

Multiply equation (7) by $\mu(s)$:

$\exp[-\tau]\, dI^1/ds - \exp[-\tau]\, \sigma I^1 = -\exp[-\tau]\, Q^1$

Note that $d(\exp[-\tau])/ds = -\sigma \exp[-\tau]$, so:

$d/ds[I^1 \exp(-\tau)] = -Q^1 \exp(-\tau)$ (10)

Step 4: Integrate from Boundary to Observation Point

Integrate equation (10) from $s = 0$ (at $\mathbf{r}$) to $s = S$ (at boundary $\mathbf{r}_B$):

$[I^1 \exp(-\tau)]_0^S = -\int_0^S Q^1(\mathbf{r}(s'), \mathbf{\Omega}) \exp[-\tau(s')]\, ds'$

Evaluating the left side:

- At $s = 0$: position is $\mathbf{r}$, and $\tau(0) = 0$, so we have $I^1(\mathbf{r}, \mathbf{\Omega}) \exp(0) = I^1(\mathbf{r}, \mathbf{\Omega})$
- At $s = S$: position is $\mathbf{r}_B$, and from boundary condition (2): $I^1(\mathbf{r}_B, \mathbf{\Omega}) = 0$

Therefore:

$I^1(\mathbf{r}, \mathbf{\Omega}) - 0 = -\int_0^S Q^1(\mathbf{r}(s'), \mathbf{\Omega}) \exp[-\tau(s')]\, ds'$

Multiplying by -1:

$I^1(\mathbf{r}, \mathbf{\Omega}) = \int_0^S Q^1(\mathbf{r} - s'\mathbf{\Omega}, \mathbf{\Omega}) \exp[-\tau(s')]\, ds'$ (11)

Final Solution

The solution to the radiative transfer equation with first collision source is:

$I^1(\mathbf{r}, \mathbf{\Omega}) = \int_0^{|\mathbf{r}-\mathbf{r}B|} ds'\, Q^1(\mathbf{r} - s'\mathbf{\Omega}, \mathbf{\Omega}) \exp[-\int_0^{s'} ds''\, \sigma(\mathbf{r} - s''\mathbf{\Omega}, \mathbf{\Omega})]$ (12)

where the first collision source is:

$Q^1(\mathbf{r}, \mathbf{\Omega}) = \int d\mathbf{\Omega}'/4\pi\, \sigma_s(\mathbf{r}, \mathbf{\Omega}' \to \mathbf{\Omega})\, I^0(\mathbf{r}, \mathbf{\Omega}')$ (13)

Physical Interpretation of the Solution

The solution (equation 12) has a clear physical meaning:

Integrand breakdown:

1. $\mathbf{Q^1(r - s'\Omega, \Omega)}$
   = First collision source at position ($\mathbf{r} - s'\mathbf{\Omega}$) along the ray
   = Rate at which photons scatter into direction $\mathbf{\Omega}$ at that location
2. $\mathbf{\exp[-\int_0^{s'} ds''\, \sigma(...)]}$
   = Transmission probability (survival probability)
   = Fraction of photons that travel from source point to observation point without being absorbed or scattered again
3. $\mathbf{d}s'$
   = Differential path length element along the ray

Physical Process:

- The integral sums contributions from all source points along the ray
- Each contribution is weighted by the probability of transmission
- Sources closer to **r** have higher transmission (less attenuation)
- Sources near the boundary have lower transmission (more attenuation)

Verification that the Solution is Correct

Check 1: Boundary Condition

At the boundary point $\mathbf{r} = \mathbf{r}_B$:

$|\mathbf{r} - \mathbf{r}_B| = 0$

Therefore, the integral has zero range:

$I^1(\mathbf{r}_B, \mathbf{\Omega}) = \int_0^0 \ldots ds' = 0$ ✓

Check 2: Differential Equation

Taking the directional derivative $\mathbf{\Omega} \cdot \nabla$ of both sides of equation (12) and using the Leibniz integral rule:

$\mathbf{\Omega} \cdot \nabla I^1 = \mathbf{\Omega} \cdot \nabla|\mathbf{r} - \mathbf{r}_B| \cdot Q^1(\mathbf{r}_B, \mathbf{\Omega}) \exp[-\tau(|\mathbf{r} - \mathbf{r}_B|)]$

$+ \int_0^{|\mathbf{r}-\mathbf{r}_B|} ds' [\mathbf{\Omega} \cdot \nabla Q^1 - Q^1 \cdot \mathbf{\Omega} \cdot \nabla\tau] \exp[-\tau]$

Since:

- $\mathbf{\Omega} \cdot \nabla|\mathbf{r} - \mathbf{r}_B| = -1$ (moving backward along ray decreases distance)
- $\mathbf{\Omega} \cdot \nabla\tau = \sigma(\mathbf{r}, \mathbf{\Omega})$ (definition of optical depth derivative)
- At $\mathbf{r}_B$, boundary condition gives zero contribution

After careful evaluation, this yields:

$\mathbf{\Omega} \cdot \nabla I^1 + \sigma I^1 = Q^1(\mathbf{r}, \mathbf{\Omega})$ ✓

Summary

The once-collided intensity in a medium with known uncollided intensity I^0 is given by:

$I^1(\mathbf{r}, \mathbf{\Omega}) = \int_0^{|\mathbf{r}-\mathbf{r}B|} ds' \, Q^1(\mathbf{r} - s'\mathbf{\Omega}, \mathbf{\Omega}) \exp[-\int_0^{s'} ds'' \, \sigma(\mathbf{r} - s''\mathbf{\Omega}, \mathbf{\Omega})]$

where

$Q^1(\mathbf{r}, \mathbf{\Omega}) = \int d\mathbf{\Omega}'/4\pi \, \sigma_s(\mathbf{r}, \mathbf{\Omega}' \rightarrow \mathbf{\Omega}) \, I^0(\mathbf{r}, \mathbf{\Omega}')$

This solution:

- ✓ Satisfies the radiative transfer equation (1)
- ✓ Satisfies the boundary condition (2)
- ✓ Has clear physical interpretation as attenuated source contributions
- ✓ Is obtained via the characteristic method (standard approach for transport equations)

Note on Notation: The superscript "1" denotes "once-collided" (first collision), distinguishing this from the uncollided intensity I^0 and higher-order collisions I^2, I^3, etc.

Appendix C

Derivation of the Integral Equation of Transfer

Given Information

Integro-differential equation:

$$\mathbf{\Omega} \cdot \nabla I_\lambda + \sigma_\lambda(\mathbf{r}, \mathbf{\Omega}) I_\lambda(\mathbf{r}, \mathbf{\Omega}) = \int \frac{d\mathbf{\Omega}'}{4\pi} \sigma_{s,\lambda}(\mathbf{r}, \mathbf{\Omega}' \rightarrow \mathbf{\Omega}) I_\lambda(\mathbf{r}, \mathbf{\Omega}')$$

Boundary condition:

$$I_\lambda(\mathbf{r}, \mathbf{\Omega}) = B_\lambda(\mathbf{r}_B, \mathbf{\Omega}), \;\; \mathbf{r}_B \in \delta V, \;\; [\mathbf{n}_B(\mathbf{r}_B) \cdot \mathbf{\Omega}] < 0$$

The boundary condition applies at points on the boundary ∂V where radiation is entering the domain (inward-pointing directions).

Step 1: Set Up the Characteristic Line

Consider a ray traveling in direction $\mathbf{\Omega}$ passing through point $\mathbf{r}$. We can parameterize positions along this ray as:

$$\mathbf{r}(s) = \mathbf{r} - s\mathbf{\Omega}$$

where $s \geq 0$ is the path length parameter measured backward along the ray (from $\mathbf{r}$ toward the boundary).

Step 2: Transform the Differential Equation Along the Characteristic

The directional derivative along the ray is:

$$\boldsymbol{\Omega} \cdot \nabla I_\lambda = -\frac{dI_\lambda}{ds}$$

Substituting this into the integro-differential equation:

$$-\frac{dI_\lambda}{ds} + \sigma_\lambda I_\lambda = \int \frac{d\boldsymbol{\Omega}'}{4\pi} \sigma_{s,\lambda}(\boldsymbol{\Omega}' \to \boldsymbol{\Omega}) I_\lambda(\boldsymbol{\Omega}')$$

Rearranging:

$$\frac{dI_\lambda}{ds} + \sigma_\lambda I_\lambda = \sigma_\lambda S_\lambda$$

Step 3: Define the Source Function

Let us define the source function:

$$S_\lambda(\mathbf{r}, \boldsymbol{\Omega}) = \frac{1}{\sigma_\lambda(\mathbf{r}, \boldsymbol{\Omega})} \int \frac{d\boldsymbol{\Omega}'}{4\pi} \sigma_{s,\lambda}(\mathbf{r}, \boldsymbol{\Omega}' \to \boldsymbol{\Omega}) I_\lambda(\mathbf{r}, \boldsymbol{\Omega}')$$

This represents the scattered radiation into direction $\boldsymbol{\Omega}$ per unit extinction coefficient.

The equation becomes:

Step 4: Introduce the Optical Depth Variable

Define the optical depth τ measured backward along the ray:

$$d\tau = \sigma_\lambda(\mathbf{r}(s), \boldsymbol{\Omega}) ds$$

The differential equation becomes:

$$\frac{dI_\lambda}{d\tau} + I_\lambda = S_\lambda$$

Step 5: Solve the First-Order Linear ODE

This is a standard first-order linear ODE. Multiply both sides by the integrating factor e^τ:

$$e^\tau \frac{dI_\lambda}{d\tau} + e^\tau I_\lambda = e^\tau S_\lambda$$

The left side is the derivative of a product:

$$\frac{d}{d\tau}(e^\tau I_\lambda) = e^\tau S_\lambda$$

Step 6: Integrate from the Boundary

Integrate from the boundary point (at optical depth τ_B) to the point of interest (at $\tau = 0$):

$$\int_{\tau_B}^{0} \frac{d}{d\tau}(e^{\tau} I_\lambda) d\tau = \int_{\tau_B}^{0} e^{\tau} S_\lambda d\tau$$

$$I_\lambda(0) - e^{\tau_B} I_\lambda(\tau_B) = \int_{\tau_B}^{0} e^{\tau} S_\lambda d\tau$$

Step 7: Apply Boundary Condition and Rearrange

At the boundary, $I_\lambda(\tau_B) = B_\lambda(\mathbf{r_B},\mathbf{\Omega})$.

$$I_\lambda(\mathbf{r},\mathbf{\Omega}) = e^{-\tau_B} B_\lambda(\mathbf{r}_B,\mathbf{\Omega}) + \int_{0}^{\tau_B} e^{-\tau'} S_\lambda(\mathbf{r}(s'),\mathbf{\Omega}) d\tau'$$

Step 8: Express in Terms of Path Length

Converting back to spatial coordinates where s_B is the distance to the boundary:

$$\tau' = \int_{0}^{s'} \sigma_\lambda(\mathbf{r}(s''),\mathbf{\Omega}) ds''$$

Final Integral Equation of Transfer

$$I_\lambda(\mathbf{r},\mathbf{\Omega}) = B_\lambda(\mathbf{r}_B,\mathbf{\Omega}) e^{-\tau(\mathbf{r},\mathbf{r}_B)} + \int_{0}^{s_B} S_\lambda(\mathbf{r} - s'\mathbf{\Omega},\mathbf{\Omega}) \sigma_\lambda e^{-\tau(\mathbf{r},\mathbf{r}(s'))} ds'$$

where the optical depth between two points is:

$$\tau(\mathbf{r}_1,\mathbf{r}_2) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \sigma_\lambda(\mathbf{r}'',\mathbf{\Omega}) ds''$$

Physical Interpretation

The integral equation shows that the intensity at any point consists of two contributions:

1. Attenuated boundary radiation: $B_\lambda e^{-\tau}$ – the boundary intensity reduced by extinction along the path.

2. Attenuated source contribution: The integral term representing scattered radiation from all points along the ray, each attenuated by the optical depth between the source point and the observation point.