Chapter 1

The Radiation Field and the Radiative Transfer Equation

1. The Radiation Field

Photons: The energy in the radiation field is assumed carried by point mass-less neutral particles called photons. The energy of a photon E (in Joules) is $\hbar\nu$, where $\hbar$ = $6.626176 \cdot 10^{-34}$ J s (Joules seconds) is Planck's constant and ν is photon frequency (in s^{-1}). Frequency is related to wavelength λ (in meters) as $\nu = c/\lambda$ where $c = 2.99792458 \cdot 10^{8}$ m s^{-1} is speed of light. Photons travel in straight lines between collisions and are regarded as a point particles, with position described in Cartesian coordinates by the vector $\underline{r} = (x, y, z)$ and direction of travel by the unit vector $\underline{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$, $\| \underline{\Omega} \| = 1$ (Fig. 1). Here and throughout the book the symbol $\| \underline{r} \|$ is used to denote the length of the vector $\underline{r}$, i.e., $\| \underline{r} \|^2 = x^2 + y^2 + z^2$. We will also use a polar coordinate system to specify the unit vector $\underline{\Omega}$. Cartesian coordinates of $\underline{\Omega}$ can be expressed via the polar angle θ and the azimuthal angle φ as $\Omega_x = \sin\theta\cos\varphi$, $\Omega_y = \sin\theta\sin\varphi$, $\Omega_z = \cos\theta$ (Fig. 1).

The description of photon distribution requires the consideration of photons traveling in directions confined to a solid angle. A solid angle is a part of space bounded by the line segment from a point (the vertex) to all points of a closed curve. A cone is an example of the solid angle which is bounded by lines from a fixed point to all points on a given circle. The solid angle represents the visual angle under which all points of the given curve can be seen from the vertex.

A measure, or "size", of a solid angle is the area of that part of the unit sphere with center at vertex that is cut off by the solid angle. Units of the solid angle are expressed in steradian (sr). For a unit sphere whose area is 4π, its solid angle is 4π sr. In the polar coordinate system, the differential solid angle $d\underline{\Omega}$ cuts an area consisting of points with polar and azimuthal angles from intervals $[\theta, \theta + d\theta]$ and $[\varphi, \varphi + d\varphi]$.

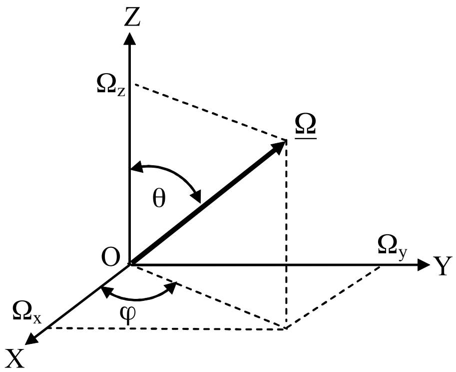

Figure 1. Representation of the unit vector $\underline{\Omega} \equiv (\Omega_x, \Omega_y, \Omega_z)$, $\| \Omega \|^2 = \Omega_x^2 + \Omega_y^2 + \Omega_z^2 = 1$, in Cartesian and polar coordinate systems. Here Ω_x, Ω_y and Ω_z are Cartesian coordinates of $\underline{\Omega}$; θ and φ are the corresponding polar and azimuthal angles in a polar coordinate system.

Particle Distribution Function: Let $f(\underline{r}, \nu, \underline{\Omega}, t)$ denote the density distribution function such that the number of photons dn at time t in the volume element $d\underline{r}$ (in m^3) about the point $\underline{r}$, with *frequency* in a frequency interval ν to $\nu + d\nu$ (in s), and traveling along a direction $\underline{\Omega}$ within solid angle $d\underline{\Omega}$ (in sr, see Problem 3) is

$$dn = f \, d\underline{r} \, d\nu \, d\underline{\Omega}. \tag{1}$$

In the *frequency domain*, the particle distribution function $f(\underline{r}, \nu, \underline{\Omega}, t)$ has units of photon number per m^3 per frequency interval per steradian (m^{-3} s sr^{-1}). In the above definition, one can use the *wavelength* interval λ to $\lambda + d\lambda$ (in m) instead of its frequency counterpart to define the particle distribution function. In the *wavelength domain*, therefore, the particle distribution function has units of photon number per m^3 per m per steradian (m^{-4} sr^{-1}).

Specific Intensity: Many radiometric devices used in remote sensing respond to radiant energy. It is convenient, therefore, to express the particle distribution in terms of energy that photons transport. Consider a volume element $d\underline{r} = d\sigma_\Omega dz$ with the base $d\sigma_\Omega$ (in m^2) perpendicular to a direction $\underline{\Omega}$ and the height dz (in m). The number of photons in this volume traveling along the direction $\underline{\Omega}$ is determined by the number of photons which cross $d\sigma_\Omega$ in the time interval t to t+dz/c where c is speed of light since the distance traversed by a photon within the interval dt = dz/c does not exceed dz. Equation (1) can be rewritten as $dn = f \, d\sigma_\Omega \, c\,dt \, d\nu \, d\underline{\Omega}$. Since the energy of one photon is $\hbar\nu$, the amount, dE, of radiant energy (in J) in a time interval dt and in the frequency interval ν to $\nu + d\nu$, which crosses a surface element $d\sigma_\Omega$ perpendicular to $\underline{\Omega}$ within a solid angle $d\underline{\Omega}$ is given by

$$dE = \hbar\nu \, dn = c\hbar\nu \, f(\underline{r}, \nu, \underline{\Omega}, t) d\sigma_\Omega dt \, d\nu \, d\underline{\Omega}. \tag{2}$$

The distribution of energy that photons transport is given by *specific intensity* (or *radiance*) defined as

$$I(\underline{r},\nu,\underline{\Omega},t) = c\hbar\nu f(\underline{r},\nu,\underline{\Omega},t). \tag{3a}$$

Its units are J $m^{-2}sr^{-1}$ in the frequency domain and J s^{-1} m^{-3} sr^{-1}=W m^{-3} sr^{-1} in the wavelength domain. Here W (1 watt = Js^{-1}) is the unit of radiant power.

Some radiometric devices count photons impinging on a detection area for a certain time interval. It is also convenient to express the photon distribution in terms of number of photons crossing a surface of unit area, per unit time per unit frequency per unit steradian. This quantity, intensity of photons (in number $s^{-1}m^{-3}sr^{-1}$), is simply the ratio between "radiant" intensity and the energy of one photon $\hbar\nu$ and can be expressed via the particle density distribution function as

$$I(\underline{r},\nu,\underline{\Omega},t) = c f(\underline{r},\nu,\underline{\Omega},t). \tag{3b}$$

We will use *intensity* as the basic radiometric quantity throughout this book, allowing for both possibilities in its definition. If the specific intensity is independent of $\underline{\Omega}$ at a point, it is said to be *isotropic* at that point. If the intensity is independent of both $\underline{r}$ and $\underline{\Omega}$, the radiation field is said to be *homogeneous* and *isotropic*. It should be emphasized that the particle distribution function f describes photons *at time* t while the specific intensity refers to radiant energy (number of photons) passing a unit area in *the time interval* t to t+dt.

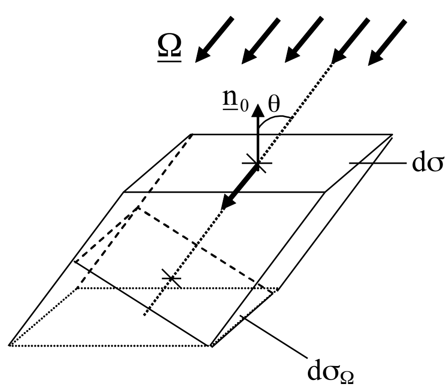

Figure 2. A beam of photons incident on the area dσ at an angle θ to the normal $\underline{n}_0$. Here $d\sigma_\Omega$ is the projection of the area dσ onto a plane perpendicular to a direction $\underline{\Omega}$ of photon travel. Its area can be expressed as $d\sigma_\Omega = d\sigma\cos\theta$. Note that $\cos\theta = |\underline{\Omega}\bullet\underline{n}_0|$ where $\underline{\Omega}\bullet\underline{n}_0$ is the scalar product of two unit vectors $\underline{\Omega}$ and $\underline{n}_0$.

Figure 2 shows an example where a photon beam of intensity I is incident on an area $d\sigma$ at an angle θ to the normal $\underline{n}_0$ to $d\sigma$. It is clear that the number of photon crossing the area dσ coincides with the number of photons crossing its projected area $d\sigma_\Omega$. Thus, the amount of radiant energy (number of photons) dE in a time interval dt, in the frequency interval ν to $\nu + d\nu$, which crosses a surface element dσ in directions confined to a differential solid angle $d\underline{\Omega}$, which is oriented at an angle θ to the normal $\underline{n}_0$ of $d\sigma$ can be expressed as

$$dE = I(\underline{r},\nu,\underline{\Omega},t)\cos\theta\, d\nu\, d\underline{\Omega}\, d\sigma\, dt. \tag{4}$$

Radiative flux density: Equation (4) gives the energy in the frequency interval ν to $\nu + d\nu$ which flows across an element area of $d\sigma$ in a direction which is inclined at an angle θ to its outward normal $\underline{n}_0$ and confined to an element of solid angle $d\underline{\Omega}$. The net flow in all direction is given by

$$F_\nu(\underline{r}) = \int_{4\pi} I(\underline{r}, \nu, \underline{\Omega}, t) \,|\, \underline{n}_0 \bullet \underline{\Omega} \,|\, d\underline{\Omega}, \tag{5}$$

where the integration is performed over the unit sphere 4π of directions. The quantity F_ν is called the *net monochromatic flux density* at $\underline{r}$ and defines the rate of flow of radiant energy across $d\sigma$ of unit area and per unit frequency interval. Its units is J m^{-2} in the frequency domain and W m^{-3} in the wavelength domain.

The net flux can in turn be represented as a sum of two hemispherical fluxes with respect to an arbitrary surface element $d\sigma$ as $F_\nu(\underline{r}) = F_\nu^+(\underline{r}) - F_\nu^-(\underline{r})$. Here $F_\nu^\pm$ are the monochromatic flux densities at different sides of $d\sigma$, or the *monochromatic irradiances*,

$$F_\nu^\pm(\underline{r}) = \int_{\mp\underline{n}_0 \bullet \underline{\Omega} > 0} I(\underline{r}, \nu, \underline{\Omega}, t) \,|\, \underline{n}_0 \bullet \underline{\Omega} \,|\, d\underline{\Omega}\,. \tag{6}$$

The *total hemispherical flux density*, in Wm^{-2}, or irradiance, for all frequencies (wavelengths) can be obtained by integrating the monochromatic irradiance over the entire electromagnetic spectrum

$$F^\pm(\underline{r}) = \int_0^\infty F_\nu^\pm(\underline{r}) d\nu\,. \tag{7}$$

The integral of the irradiance over an area A is the total flux, in W, or *radiant power*,

$$F^\pm = \int_A F^\pm(\underline{r}) d\sigma\,. \tag{8}$$

For homogeneous and isotropic radiation, intensity $I(\underline{r}, \nu, \underline{\Omega}, t) = i(\nu, t)$ is independent of angular and spatial variables, the above quantities are

$$F_\nu^\pm(\underline{r}) = \pi i(\nu, t), \quad F^\pm(\underline{r}) = \pi \int_0^\infty i(\nu, t) d\nu, \quad F^\pm = \pi A \int_0^\infty i(\nu, t) d\nu\,. \tag{9}$$

Normalization of the above quantities by the energy of one photon $\hbar\nu$ results in corresponding fluxes for photons.

2. Interaction of Radiation with Matter

Absorption: The *absorption coefficient* σ_a (in m^{-1}) is defined such that the probability of a photon being absorbed while traveling a distance ds is $\sigma_a(\underline{r}, \nu, \underline{\Omega}, t)ds$. An absorption event signifies true loss of a photon from the count.

Scattering: The *scattering coefficient* σ'_s (in m^{-1}) is defined in analogy to the absorption coefficient,

$$\text{Probability of scattering} = \sigma'_s(\underline{r}, \nu, \underline{\Omega}, t)\ ds.$$

Unlike absorption, a scattering event serves to change the direction and/or frequency of the incident photon. Thus, it is convenient to define a *differential scattering coefficient* σ_s (in m^{-1} sr^{-1}) as,

$$\text{Probability of scattering} = \sigma_s(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t)\ ds\ d\underline{\Omega}.$$

The change in photon frequency as a result of a scattering event is generally not relevant in optical remote sensing of vegetation. It is important to note that photon scattering in vegetation media depends on the coordinates of $\underline{\Omega}'$ and $\underline{\Omega}$ in general. The scattering coefficients σ'_s and σ_s are related as

$$\sigma'_s(\underline{r}, \nu, \underline{\Omega}, t) = \int_0^\infty d\nu'' \int_{4\pi} d\underline{\Omega}''\, \sigma_s(\underline{r}, \nu \to \nu'', \underline{\Omega} \to \underline{\Omega}'', t). \quad (10)$$

In some cases, the differential scattering coefficient is decomposed into the product

$$\sigma_s(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t) = \sigma'_s(\underline{r}, \nu', \underline{\Omega}', t)\, K(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t), \quad (11)$$

such that, the kernel K, termed a s*cattering phase function*, has the interpretation of a probability density function,

$$\int_0^\infty d\nu \int_{4\pi} d\underline{\Omega}\, K(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t) = 1. \quad (12)$$

In the case of coherent scattering, there is no frequency change upon scattering and

$$K(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t) = K(\underline{r}, \underline{\Omega}' \to \underline{\Omega}, t)\, \delta(\nu' - \nu), \quad (13a)$$

where δ is the Dirac delta function. In the case of isotropic scattering,

$$K(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t) = \frac{1}{4\pi} K(\underline{r}, \nu' \to \nu, t). \quad (13b)$$

Therefore, the simplest scattering kernel corresponds to isotropic coherent scattering, namely,

$$K(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t) = \frac{1}{4\pi} \delta(\nu' - \nu). \tag{13c}$$

Extinction: The *extinction* or the *total interaction coefficient* σ (in m^{-1}) is simply the sum $\sigma_a + \sigma_s'$. Therefore, $\sigma(\underline{r}, \nu, \underline{\Omega}, t)\,ds$ is the probability that a photon would disappear from the beam while traveling a distance ds in the medium (note that it can reappear at a different frequency and/or direction.) The quantity $1/\sigma$ denotes *photon mean free path*, that is, the average distance a photon will travel in the medium before suffering a collision. The dependence on the direction of photon travel is noteworthy and is especially important in the case of vegetation media.

Single Scattering Albedo: The probability of scattering given that a collision has occurred is given by the *single scattering albedo*, $\omega = \sigma_s'/\sigma$ (dimensionless). In the case of conservative scattering, $\omega = 1$. The case $\omega = 0$ corresponds to pure absorption.

Emission: Photons can be introduced into the medium through external and/or internal sources. In the frequency domain, the number of photons emitted by volume dr at r in the direction $\underline{\Omega}$ about the differential solid angle $d\underline{\Omega}$ at frequency ν in the interval ν to $\nu + d\nu$ between t and t+dt is $q(\underline{r}, \nu, \underline{\Omega}, t)d\underline{r}d\nu\, d\underline{\Omega}\, dt$.

It should be noted that we neglect photon to photon interaction in the above definitions. This means that the photon density is low, that is, low enough such that the overlap in the tails of wavepackets of two photons is negligibly small. This is especially required in the case of source photons emitted at the same location. We also assume that collisions and emission processes occur instantaneously. This imposes a limit on the time resolution over which the above definitions are applicable.

3. The Equation of Transfer

Consider the change dN in time Δt of the number of photons which are located in a volume element $d\underline{r} = \Delta\underline{S}\Delta\xi$ about the point $\underline{r}$ (Fig. 3). Here the base $\Delta\underline{S}$ of the volume dr is perpendicular to the direction $\underline{\Omega}$ of photon travel and the height $\Delta\xi$=cΔt where c is speed of light. The number of photons in this volume traveling along the direction $\underline{\Omega}$ is determined by the number of photons which cross ΔS in the time interval t to t+dz/c. It follows from Eq. (3b) that this count is given by

$$\frac{1}{c} I(\underline{r}_1, \nu, \underline{\Omega}, t)\, \Delta\underline{S}\, \Delta\xi\, \Delta\nu\, \Delta\underline{\Omega}.$$

The number of photons leaving the volume $d\underline{r}$ through its lower surface ΔS in the direction $\underline{\Omega}$ in the time interval t to t+dz/c can be expressed as

$$\frac{1}{c} I(\underline{r}_2 + c\Delta t\underline{\Omega}, \nu, \underline{\Omega}, t + \Delta t)\ \Delta\underline{S}\ \Delta\xi\ \Delta\nu\ \Delta\underline{\Omega}.$$

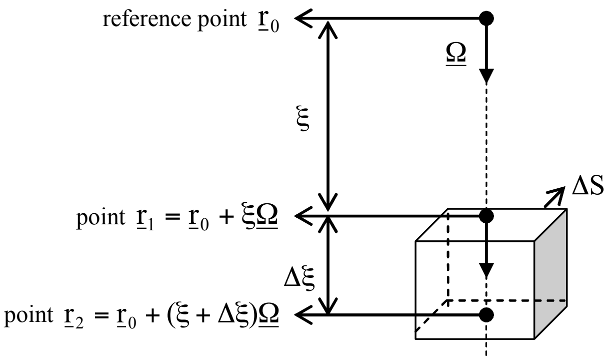

Figure 3. A volume element $d\underline{r} = \Delta\underline{S}\Delta\xi$ with the base $\Delta\underline{S}$ perpendicular to a direction $\underline{\Omega}$ and the height $\Delta\xi = c\Delta t$. Points $\underline{r}_1$ and $\underline{r}_2$ on the upper and lower boundaries of the volume element can be represented as $\underline{r}_1 = \underline{r}_0 + \xi\underline{\Omega}$ and $\underline{r}_2 = \underline{r}_0 + (\xi + \Delta\xi)\underline{\Omega}$, respectively. Here ξ and $\xi + \Delta\xi$ are distances between these points and a point $\underline{r}_B$ on the boundary δV along a direction opposite to $\underline{\Omega}$.

The change in time Δt of the number of photons in $d\underline{r}$ is

$$dN = \frac{1}{c}\Delta I\,\Delta\underline{S}\ \Delta\xi\ \Delta\nu\ \Delta\underline{\Omega},$$

where

$$\Delta I = I(\underline{r} + c\Delta t\Omega, \nu, \underline{\Omega}, t + \Delta t) - I(\underline{r}, \nu, \underline{\Omega}, t). \tag{14}$$

In the increment (14), the spatial variable $\underline{r}$ depends on t. Using the chain rule for function of several variables, one gets

$$\Delta I = \frac{\partial I}{\partial t}\Delta t + \frac{\partial I}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial I}{\partial y}\frac{dy}{dt}\Delta t + \frac{\partial I}{\partial z}\frac{dz}{dt}\Delta t,$$

where x, y and z are Cartesian coordinates of points $\underline{r} + ct\underline{\Omega}$, $0 \le t \le \xi/c$. Thus,

$$dN = \left(\frac{1}{c}\frac{\partial I}{\partial t} + \Omega_x\frac{\partial I}{\partial x} + \Omega_y\frac{\partial I}{\partial y} + \Omega_z\frac{\partial I}{\partial z}\right)\Delta t\,\Delta\underline{S}\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega},$$

where Ω_x, Ω_y, and Ω_z are Cartesian coordinates of the unit vector $\underline{\Omega}$. The first term, $\partial I/\partial t$, in parentheses is the time rate of change of the number of photons. The other terms represent a derivative $\underline{\Omega} \bullet \nabla I$ at $\underline{r}$ along the direction $\underline{\Omega}$ which shows the net rate of photons streaming out of the volume element along the direction $\underline{\Omega}$. Thus,

$$dN = \underbrace{\frac{1}{c}\frac{\partial I}{\partial t}\Delta t\,\Delta \underline{S}\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}}_{\text{temporal rate of change}} + \underbrace{(\underline{\Omega}\bullet\nabla I)\,\Delta t\,\Delta\underline{S}\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}}_{\text{streaming}}. \tag{15}$$

Here ∇ is the vector operator, called "nabla." Given a scalar function f, vector ∇f has the form

$$\nabla I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}, \frac{\partial I}{\partial z}\right).$$

The change described by Eq. (15) is due to four processes – absorption, outscattering, inscattering and emission, and these are described below.

Absorption: A fraction of photons in the volume element $d\underline{r}$ will be absorbed while traveling a distance $\Delta\xi = c\,\Delta t$ along the direction $\underline{\Omega}$. This fraction is determined by the probability $\sigma_a \Delta\xi$. Thus, the number of absorbed photons is

$$\begin{aligned}\text{absorption} &= \underbrace{\frac{1}{c} I(\underline{r},\nu,\underline{\Omega},t)\,\Delta\underline{S}\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}}_{\text{number of photons}}\;\underbrace{\sigma_a(\underline{r},\nu,\underline{\Omega},t)\Delta\xi}_{\substack{\text{probabiliy of absorption}\\ \text{while traveling } \Delta\xi=c\Delta t}} \\ &= \sigma_a(\underline{r},\nu,\underline{\Omega},t)\; I(\underline{r},\nu,\underline{\Omega},t)\;\Delta\underline{S}\,\Delta t\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}.\end{aligned} \tag{16}$$

Outscattering: Another fraction of photons in the volume element $\underline{r}$ traveling in the direction $\underline{\Omega}$ will change their direction and/or frequency as a result of interaction with matter. The number of photons "lost" due to *outscattering* from ν, $\underline{\Omega}$ to all other frequencies and directions while traveling a distance $\Delta\xi = c\,\Delta t$ is given by

$$\begin{aligned}\text{outscattering} &= c\;\Delta t\,\Delta\underline{S}\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}\int_0^\infty d\nu' \int_{4\pi} d\underline{\Omega}'\;\sigma_s(\underline{r},\nu\to\nu',\underline{\Omega}\to\underline{\Omega}',t)\;\frac{1}{c}\,I(\underline{r},\nu,\underline{\Omega},t) \\ &= \sigma_s'(\underline{r},\nu,\underline{\Omega},t)\;I(\underline{r},\nu,\underline{\Omega},t)\Delta t\,\Delta\underline{S}\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}.\end{aligned} \tag{17}$$

Inscattering: Similarly, the number of photons gained due to *inscattering* to ν, $\underline{\Omega}$ from all other frequencies and directions can be evaluated as

$$\text{inscattering} = \Delta t\,\Delta\underline{S}\;\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}\int_0^\infty d\nu' \int_{4\pi} d\underline{\Omega}'\;\sigma_s(\underline{r},\nu'\to\nu,\underline{\Omega}'\to\underline{\Omega},t)\;I(\underline{r},\nu',\underline{\Omega},t). \tag{18}$$

The rate of the production of photons in the volume element is simply

$$\text{emission} = c\,q(\underline{r},\nu,\underline{\Omega},t)\;\Delta t\,\Delta\underline{S}\,\Delta\xi\,\Delta\nu\,\Delta\underline{\Omega}. \tag{19}$$

Transfer Equation: The equation of transfer is essentially a statement of photon number conservation at $\underline{r}$ by equating the sum of the four terms, Eqs. (16) to (19), with appropriate signs to designate a loss or gain, to the overall rate of change given by Eq. (15):

dN= - absorption - outscattering + inscattering + emission,

or, after dividing all terms by $c\Delta t \Delta S \Delta\xi \Delta\nu \Delta\underline{\Omega}$ and accounting for the definition of the extinction coefficient $\sigma = \sigma_a + \sigma_s'$, one gets

$$\frac{1}{c}\frac{\partial I}{\partial t} + (\underline{\Omega} \bullet \nabla I) + \sigma(\underline{r}, \nu, \underline{\Omega}, t)\, I(\underline{r}, \nu, \underline{\Omega}, t)$$
$$= \int_0^\infty d\nu' \int_{4\pi} d\underline{\Omega}'\, \sigma_s(\underline{r}, \nu' \to \nu, \underline{\Omega}' \to \underline{\Omega}, t)\, I(\underline{r}, \nu', \underline{\Omega}', t) + q(\underline{r}, \nu, \underline{\Omega}, t). \qquad (20)$$

Sometimes, the second and the third terms on the left hand side of Eq. (20) are grouped together; the term $[\underline{\Omega} \bullet \nabla + \sigma]$ then denotes the *streaming-collision* operator. An equation for the particle density distribution function can be obtained by normalizing Eq. (20) by $c\hbar\nu$ if I is the radiant intensity or by c if I represents intensity of photons (see Eqs. (3a) and (3b)).

It should be noted that this equation gives the expected or mean value of the photon distribution. Fluctuations about the mean are not considered. The derived equation also assumes unpolarized light. Four parameters are required to specify the state of polarization of a beam of light, and accordingly, a proper description of photon transport including polarization effects involves four coupled equations of transfer. Assuming the light to be unpolarized by the medium, these equations can be averaged to derive a single equation of transfer and this involves some error. Finally, the radiative transfer equation (20) does not describe behavior resulting from interference of waves. Therefore, the equation is valid only when the distance between scatterers is large compared to the wave packets.

4. Initial and Boundary Conditions

In many practical cases, one is interested in the photon distribution in a restricted region of space. It is necessary to specify a domain V in which the radiative transfer process is studied and a surface δV that bounds V. Equation (20) is usually formulated for a domain V whose composition and shape depends on a specific problem under consideration. In solving the radiative transfer equation, it is necessary to specify both the photon distribution in V at some initial time $t = 0$ (initial condition) and the photon distribution incident on V at all times (boundary condition). The initial condition is given by

$$I(\underline{r}, \nu, \underline{\Omega}, 0) = I_0(\underline{r}, \nu, \underline{\Omega}), \quad \underline{r} \in V. \qquad (21)$$

The boundary condition specifies the radiation entering the domain V through points on the boundary δV,

$$I(\underline{r}_B, \nu, \underline{\Omega}, t) = B(\underline{r}_B, \nu, \underline{\Omega}, t), \quad \underline{r}_B \in \delta V, \quad \underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0. \tag{22}$$

Here B is the *intensity* of photons incident on the domain V at point $\underline{r}_B$ on the surface δV; $\underline{n}(\underline{r}_B)$ is an outward normal vector at this point (Fig. 4). The radiative transfer problem is thus completely specified by the equation of transfer [Eq. (20)], the initial condition [Eq. (21)] and the boundary condition [Eq. (22)].

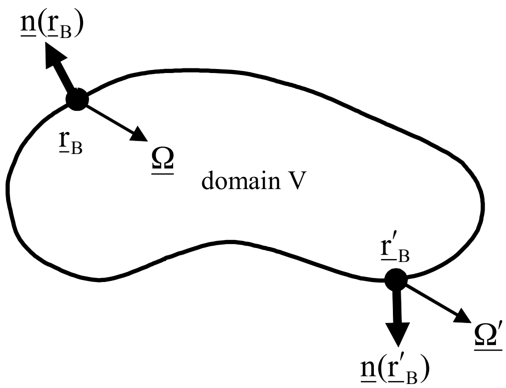

Figure 4. Directions $\underline{\Omega}$ ($\underline{\Omega}'$) along which incident photons can enter (exit) the domain V through the point $\underline{r}_B$ ($\underline{r}'_B$) on the boundary δV satisfies the inequality $\underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0$ ($\underline{n}(\underline{r}'_B) \bullet \underline{\Omega}' > 0$). Here $\underline{n}(\underline{r}_B)$($\underline{n}(\underline{r}'_B)$) is the outward normal at $\underline{r}_B$ ($\underline{r}'_B$).

The incoming radiation B can result from sources on the boundary δV and photons from V incident on the boundary that the boundary reflects back to the domain V. In the case of the boundary coherent scattering, the incoming radiation B can be written as

$$B(\underline{r}_B, \nu, \underline{\Omega}, t) = \frac{1}{\pi} \int_{\delta V} d\underline{r}'_B \int_{\underline{\Omega}' \bullet \underline{n}(\underline{r}_B) > 0} \rho_B(\underline{r}'_B, \underline{\Omega}'; \underline{r}_B, \underline{\Omega}) \, | \underline{n}(\underline{r}'_B) \bullet \underline{\Omega}' | \, I(\underline{r}'_B, \nu, \underline{\Omega}', t) d\underline{\Omega}'$$

$$+ q_B(\underline{r}_B, \underline{\Omega}, t), \quad \underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0. \tag{23}$$

Here $\rho_B(\underline{r}'_B, \underline{\Omega}'; \underline{r}'_B, \underline{\Omega})$ is the *boundary scattering function*; that is, the probability density that a photon having escaped from the domain V through the point $\underline{r}'_B \in \delta V$ in the direction $\underline{\Omega}'$ will come back to V through the point $\underline{r}_B \in \delta V$ in the direction $\underline{\Omega}$. It should be emphasized that in general the boundary condition depends on the solution I of the radiative transfer problem. The case of *vacuum boundary condition* refers to $\rho_B = 0$ and $q_B = 0$.

5. Stationary Radiative Transfer Problem

If the extinction and differential scattering coefficients, emission and the boundary condition do not change with time, $\partial I / \partial t = 0$, the radiative transfer problem becomes a stationary radiative transfer problem. In the case of coherent scattering, the *boundary value problem for radiative transfer equation* in the *wavelength domain* has the form

$$\underline{\Omega} \bullet \nabla I_\lambda(\underline{r},\underline{\Omega}) + \sigma_\lambda(\underline{r},\underline{\Omega})\, I_\lambda(\underline{r},\underline{\Omega}) = \int_{4\pi} d\underline{\Omega}'\, \sigma_{s,\lambda}(\underline{r},\underline{\Omega}' \to \underline{\Omega})\, I_\lambda(\underline{r},\underline{\Omega}') + q_\lambda(\underline{r},\underline{\Omega}), \qquad (24a)$$

$$I_\lambda(\underline{r}_B,\underline{\Omega}) = B_\lambda(\underline{r}_B,\underline{\Omega}), \quad \underline{r}_B \in \delta V, \quad \underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0. \qquad (24b)$$

Here $I_\lambda(\underline{r},\underline{\Omega})$ is the monochromatic specific intensity which depend on wavelength λ, location r and direction $\underline{\Omega}$. Note that the wavelength λ is a parameter of the radiative transfer problem. We have emphasized this feature in notations by moving the wavelength from the argument list to subscript. In our analyses of the boundary value problem, therefore, we will often suppress this parameter in notations. The stationary radiative transfer equation is the basic tool used in optical remote sensing. This equation, notations and the convention regarding the wavelength dependence introduced here will be used in the rest of this book.

6. Green's Function and the Reciprocity Principle

Consider a medium V bounded by a non-reflecting ($\rho_B = 0$) and non-emitting ($q_B = 0$) boundary δV. The *volume Green's function*, $G_V(\underline{r},\underline{\Omega};\underline{r}',\underline{\Omega}')$, is the radiative response of V at a point $\underline{r}$, in direction $\underline{\Omega}$, to a monodirectional point source located at a given point $\underline{r}'$, continuously emitting photons in a given direction $\underline{\Omega}'$. The *volume Green* function satisfies the stationary radiative transfer equation [Eq. (24a)] with a delta function source term $q(\underline{r},\underline{\Omega}) = \delta(\underline{\Omega} - \underline{\Omega}')\,\delta_V(\underline{r} - \underline{r}')$ located at $\underline{r}'$ and with zero incoming radiation (B=0), that is,

$$\underline{\Omega} \bullet \nabla G_V(\underline{r},\underline{\Omega};\underline{r}',\underline{\Omega}') + \sigma(\underline{r},\underline{\Omega})\, G_V(\underline{r},\underline{\Omega};\underline{r}',\underline{\Omega}')$$

$$= \int_{4\pi} \sigma_s(\underline{r},\underline{\Omega}'' \to \underline{\Omega}) G_V(\underline{r},\underline{\Omega}'';\underline{r}',\underline{\Omega}') d\underline{\Omega}'' + \delta(\underline{\Omega} - \underline{\Omega}')\delta_V(\underline{r} - \underline{r}'), \qquad (25a)$$

$$G_V(\underline{r}_B,\underline{\Omega};\underline{r}',\underline{\Omega}') = 0, \quad \underline{r}_B \in \delta V, \quad \underline{\Omega} \bullet \underline{n}(\underline{r}_B) < 0. \qquad (25b)$$

Here $\delta(\underline{\Omega} - \underline{\Omega}')$, in sr^{-1}, and $\delta_V(\underline{r} - \underline{r}')$, in m^{-3}, are Dirac delta functions. Note that $\delta(\underline{\Omega} - \underline{\Omega}')\delta_V(\underline{r} - \underline{r}')$ is a *volume source* normalized by its power. The volume Green function, therefore, is expressed in $m^{-2}sr^{-1}$. It should be also noted that the point $\underline{r}'$ and the direction $\underline{\Omega}'$ of the monodirectional source are parameters in the radiative transfer equation; that is, the determination of the complete Green function requires the solution of Eq. (25) for every point $\underline{r}'$ from V and the direction $\underline{\Omega}'$.

The *surface Green's function*, $G_S(\underline{r},\underline{\Omega};\underline{r}'_B,\underline{\Omega}')$, is the solution to the transport equation with the source $q(\underline{r},\underline{\Omega}) = 0$ and the boundary condition

$$G_S(\underline{r},\underline{\Omega};\underline{r}'_B,\underline{\Omega}') = \delta(\underline{\Omega} - \underline{\Omega}')\delta_S(\underline{r}_B,\underline{r}'_B), \quad \underline{r}'_B \in \delta V, \quad \underline{n}(\underline{r}'_B) \bullet \underline{\Omega}' < 0. \qquad (26)$$

Here $\delta_S(\underline{r}_B, \underline{r}'_B)$ is a two-dimensional delta function (in m^{-2}). Because the volume sources can be located on the boundary, the volume and surface Green functions are related [Case and Zweifel, 1970]

$$G_S(\underline{r}, \underline{\Omega}; \underline{r}'_B, \underline{\Omega}') = |\,\underline{n}(\underline{r}'_B) \bullet \underline{\Omega}'\,|\, G_V(\underline{r}, \underline{\Omega}; \underline{r}'_B, \underline{\Omega}'). \tag{27}$$

In terms of these two Green's functions, we may write the general solution to the transport equation with arbitrary source $q(\underline{r}, \underline{\Omega})$ and boundary conditions with sources q_B on the non-reflecting boundary δV ($\rho_B = 0$) as

$$I(\underline{r}, \underline{\Omega}) = \int_V d\underline{r}' \int_{4\pi} G_V(\underline{r}, \underline{\Omega}; \underline{r}', \underline{\Omega}')\, q(\underline{r}', \underline{\Omega}')\, d\underline{\Omega}' + \int_{\delta V} dS \int_{\underline{n}(\underline{r}'_B) \bullet \Omega' < 0} d\underline{\Omega}'\, G_S(\underline{r}, \underline{\Omega}; \underline{r}'_B, \underline{\Omega}')\, q_B(\underline{r}'_B, \underline{\Omega}'). \tag{28}$$

The first term in Eq. (28) is the solution of the radiative transfer equation with the internal source $q(\underline{r}, \underline{\Omega})$ and no incoming radiance. The second term describes the 3D radiation field in V generated by sources q_B distributed over the non-reflecting boundary δV.

Let the differential scattering coefficient satisfies the symmetry property $\sigma_s(\underline{r}, \underline{\Omega}' \to \underline{\Omega}) = \sigma_s(\underline{r}, -\underline{\Omega} \to -\underline{\Omega}')$. Under this condition, the volume Green's function possesses the following symmetry property

$$G_V(\underline{r}, \underline{\Omega}; \underline{r}', \underline{\Omega}') = G_V(\underline{r}', -\underline{\Omega}'; \underline{r}, -\underline{\Omega}). \tag{29}$$

This equality expresses the fundamental reciprocity theorem for the stationary radiative transfer equation: *the intensity* $I(\underline{r}, \underline{\Omega})$ *at* $\underline{r}$ *in the direction* $\underline{\Omega}$ *due to a point source at* $\underline{r}'$ *emitting in direction* $\underline{\Omega}'$ *is the same as the intensity* $I(\underline{r}', -\underline{\Omega}')$ *at* $\underline{r}'$ *in the direction* $-\underline{\Omega}'$ *due to a point source at* $\underline{r}$ *emitting in direction* $-\underline{\Omega}$.

The Green function concept was originally developed in neutron transport theory [Bell and Glasstone, 1970]. It has enabled the reformulation of the radiative transfer problems in terms of some "basic" sub-problems and to express the solution of the transport equation with arbitrary sources and boundary conditions as a superposition of the solutions of the basic sub-problems. We will demonstrate this technique later with a relevant example for radiative transfer in the canopy-surface-atmosphere system.

7. Operator Notations.

We introduce the *streaming-collision*, L, and *scattering*, S, operators as

$$\mathrm{LI} = \underline{\Omega} \bullet \nabla \mathrm{I}(\underline{r},\underline{\Omega}) + \sigma(\underline{r},\underline{\Omega})\mathrm{I}(\underline{r},\underline{\Omega}), \quad \mathrm{SI} = \int_{4\pi} \sigma_s(\underline{r},\underline{\Omega}' \to \underline{\Omega})\mathrm{I}(\underline{r},\underline{\Omega}')d\underline{\Omega}'. \tag{30}$$

We will use the notation $I^{\pm}(\underline{r}_B,\underline{\Omega})$ to denote the intensity of radiation exiting (sign "+") or entering (sign "–") the domain V through the point $\underline{r}_B$, i.e., $I^{\pm}(\underline{r}_B,\underline{\Omega})$ gives values of the intensity at points $\underline{r}_B$ on the boundary δV in directions satisfying the inequality $\underline{n}(\underline{r}_B) \bullet \underline{\Omega} > 0$. To describe reflective properties of the boundary δV, a scattering operator defined on the boundary δV for the intensity I^+ of medium leaving radiation is introduced as

$$\mathcal{R}I^+ = \frac{1}{\pi}\int_{\delta V} d\underline{r}'_B \int_{\underline{n}(\underline{r}'_B)\bullet\underline{\Omega}'>0} \rho_B(\underline{r}'_B,\underline{\Omega}';\underline{r}_B,\underline{\Omega})\,|\,\underline{n}(\underline{r}'_B)\bullet\underline{\Omega}'\,|\,I^+(\underline{r}'_B,\underline{\Omega}')d\underline{\Omega}'. \tag{31}$$

In terms of these notations, the boundary value problem Eq. (24) for the three-dimensional stationary radiative transfer equation can be expressed as

$$\mathrm{LI} = \mathrm{SI} + q, \quad I^- = \mathcal{R}I^+ + q_B. \tag{32}$$

The boundary value problem is said to be the *standard problem* if $\mathcal{R} = 0$ and $q_B = 0$. To emphasize this in notations, we will use symbol L_0 to denote the streaming-collision operator corresponding to the standard problem.

The boundary value problem for a domain with non-reflecting boundary can always be reduced to a standard problem. Indeed, the solution to the boundary value problem $\mathrm{LI} = \mathrm{SI} + q$, $I^- = q_B$ can be represented as the sum of two components, $I = Q + I_{dif}$. The first term describes intensity of radiation generated by *uncollided photons*; that is, photons from the boundary source q_B that have not undergone interactions within the domain V. It satisfies the equation $LQ = 0$ and the boundary condition $Q^- = q_B$. The second term, I_{dif}, describes a *collided*, or *diffuse*, radiation field; that is, radiation field generated by photons scattered one or more times. It satisfies the standard problem $L_0 I_{dif} = SI_{dif} + q'$ with the volume source $q' = q + SQ$. The uncollided component Q acts as a source term for scattering process and thus the term SQ gives intensity of photons from the uncollided field Q just after their first scattering event.

In mathematical literature, the standard problem is often formulated in functional spaces. The theory of functional analysis [Riesz and B. Sz.-Nagy 1990; Kantorovich and Akilov, 1964; Krein, 1972] requires the specification of two sets. The first – the domain D of the operator L_0 – identifies "possible candidates" for the solution. The second space – the range H of the operator L_0 – specifies mathematical properties of "acceptable" volume sources. Vladimirov [1963] provides a full mathematical description of the standard problem for the following family of functional spaces. The range H_p, $0 < p \le \infty$, consists of functions $q(\underline{r},\underline{\Omega})$ for which the norm $\|q\|_p$ exists, i.e.,

$$\| q \|_p = \int_{4\pi} d\underline{\Omega} \int_V d\underline{r} \sigma(\underline{r},\underline{\Omega}) \, | q(\underline{r},\underline{\Omega}) |^p < \infty,$$
$$0 < p < \infty,$$
$$\| q \|_\infty = \operatorname*{supvrai}_{\underline{r}\in V, \underline{\Omega}\in 4\pi} | q(\underline{r},\underline{\Omega}) | < \infty. \quad (33)$$

The set D_p, $0 < p \le \infty$, includes all functions (1) which satisfy the zero boundary condition (I=0) and (2) whose transformations L_0I and SI exist and are elements of the space H_p. The standard problem is formulated as follows: given q from H_p find an element I from D_p for which $L_0 I = SI + q$. "Visually," this formulation is similar to problems in linear algebra, i.e., where L_0 and S are matrixes, I and q are vectors, and the norm Eq. [33] is the length of the vector. Vladimirov [1963] showed that such an interpretation of the standard problem, with some caveats, is valid and many results from matrix theory can be applied to the radiative transfer equation. This level of abstraction helps to derive many practically important properties of the radiation field whose direct derivation is either very difficult or impossible. We will demonstrate this technique with a relevant example for canopy spectral response to incident solar radiation.

8. The Equation of Transfer in Integral Form

The standard problem can be transformed to two types of *integral equations*. The first one is obtained by inverting the streaming-collision operator L_0, i.e., $I = L_0^{-1} SI + L_0^{-1} q$. The second equation is formulated for a source function J defined as $J = SI + q$. It follows from Eq. (30) with $\mathcal{R} = 0$ and $q_B = 0$ that the intensity I and source function J are related as $I = L_0^{-1} J$. Substituting this equation into the definition of J results in an operator equation of the form $J = SL_0^{-1} J + q$.

Both integral equations require the specification of the inverse operator L_0^{-1} which acts either on SI or J. Let u represents either SI or J. The function $v = L_0^{-1} u$ satisfies the equation

$$\underline{\Omega} \bullet \nabla v(\underline{r},\underline{\Omega}) + \sigma(\underline{r},\underline{\Omega}) \, v(\underline{r},\underline{\Omega}) = u(\underline{r},\underline{\Omega}), \quad (34)$$

with the zero boundary condition, i.e., $u(\underline{r}_B,\underline{\Omega}) = 0$, $\underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0$. For a straight line $\underline{r}_B + \eta\underline{\Omega}$, $-\infty < \eta < \infty$, along an incoming direction $\underline{\Omega}$, $\underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0$, this equation takes the following form

$$\frac{dv(\underline{r}_B + \xi\underline{\Omega},\underline{\Omega})}{d\xi} + \sigma(\underline{r}_B + \xi\underline{\Omega},\underline{\Omega}) \, v(\underline{r}_B + \xi\underline{\Omega},\underline{\Omega}) = u(\underline{r}_B + \xi\underline{\Omega},\underline{\Omega}), \quad v(\underline{r}_B,\Omega) = 0. \quad (35)$$

This is an ordinary differential with respect to ξ. Its solution is

$$v(\underline{r}_B + \xi\underline{\Omega}, \underline{\Omega}) = \int_{4\pi}\int_0^{\xi} \exp\left(-\int_{\xi'}^{\xi} d\xi'' \, \sigma(\underline{r}_B + \xi''\underline{\Omega}', \underline{\Omega}') \right) u(\underline{r}_B + \xi'\underline{\Omega}', \underline{\Omega}') \, \delta(\underline{\Omega} - \underline{\Omega}') d\underline{\Omega}' d\xi' \,. \qquad (36)$$

Note that we have artificially expressed the solution of the ordinary integral equation as an integral over $d\xi'\underline{\Omega}'$. The presence of the delta function $\delta(\underline{\Omega} - \underline{\Omega}')$ in Eq. (36), however, makes this integral equivalent to an integral over the line $\underline{r}_B + \eta\underline{\Omega}$ along the direction $\underline{\Omega}$ which is directly obtainable from Eq. (35). Let $\underline{r}$ and $\underline{r}' = \underline{r} - \xi'\underline{\Omega}'$, $\xi' \geq 0$, be two points on the line $\underline{r}_B + \eta\underline{\Omega}'$. We make use of the relationship $d\underline{r}' = \xi'^2 d\xi' d\underline{\Omega}'$ to convert the volume element $\xi'^2 d\xi' d\Omega'$ expressed in polar coordinates with the origin at $\underline{r}$ into the volume element $d\underline{r}'$ in Cartesian coordinates. Noting that $\|\underline{r} - \underline{r}'\| = \xi'$, one can express the unit vector $\underline{\Omega}'$ as $\underline{\Omega}' = (\underline{r} - \underline{r}')/\|\underline{r} - \underline{r}'\|$. In Cartesian coordinates, the function $v = L_0^{-1}u$ can be rewritten as

$$L_0^{-1}u = v(\underline{r}, \underline{\Omega}) = \int_V \frac{\exp[-\tau(\underline{r}, \underline{r}', \underline{\Omega})]}{\| \underline{r} - \underline{r}' \|^2} u(\underline{r}', \underline{\Omega}) \, \delta\left(\underline{\Omega} - \frac{\underline{r} - \underline{r}'}{\| \underline{r} - \underline{r}' \|} \right) d\underline{r}' \qquad (37)$$

Here $\tau(\underline{r}, \underline{r}', \underline{\Omega})$ is the optical distance between points $\underline{r}$ and $\underline{r}'$ on a straight line along the direction $\underline{\Omega}$, i.e.,

$$\tau(\underline{r}, \underline{r}', \underline{\Omega}) = \int_0^{\|\underline{r} - \underline{r}'\|} d\xi'' \, \sigma(\underline{r} - \xi''\underline{\Omega}, \underline{\Omega}) \,. \qquad (38)$$

The δ-function in Eq. (37) indicates that the points $\underline{r}$ and $\underline{r}'$ lie on a line along the direction $\underline{\Omega}$. Equation (37) specifies the operator L_0^{-1} which sets in correspondence to a volume source u the three dimensional distribution $v(\underline{r}, \underline{\Omega})$ of photons from the source u that arrive at point r along the direction $\underline{\Omega}$ without suffering a collision.

Substituting $u = SI$ into Eq. (37) one obtains the following integral equation

$$I(\underline{r}, \underline{\Omega}) = \int_{4\pi}\int_V \mathcal{K}_I(\underline{r}', \underline{\Omega}' \to \underline{r}, \underline{\Omega}) I(\underline{r}', \underline{\Omega}') d\underline{r}' d\underline{\Omega}' + Q(\underline{r}, \underline{\Omega}) \,. \qquad (39)$$

Here

$$\mathcal{K}_I(r', \underline{\Omega}' \to \underline{r}, \underline{\Omega}) = \frac{\exp[-\tau(\underline{r}, \underline{r}', \underline{\Omega})]}{\| \underline{r} - \underline{r}' \|^2} \sigma_s(\underline{r}', \Omega' \to \underline{\Omega}) \, \delta\left(\underline{\Omega} - \frac{\underline{r} - \underline{r}'}{\| \underline{r} - \underline{r}' \|} \right), \qquad (40)$$

and the source $Q = L_0^{-1}q$ is calculated using Eq. (37). The kernel $\mathcal{K}_I$ is the transition density, i.e., $\mathcal{K}_I d\underline{r}' d\underline{\Omega}'$ is the probability that photons which have undergone interactions at r′ in the direction $\underline{\Omega}'$ will have their next interaction at $\underline{r}$ along the direction $\underline{\Omega}$.

Multiplying Eq. (37) by the differential scattering coefficient $\sigma_s(\underline{r}, \underline{\Omega}' \to \underline{\Omega})$ and integrating over all scattering directions $\underline{\Omega}'$ one obtains a kernel $\mathcal{K}_S$ of the integral operator SL_0^{-1}:

$$\mathcal{K}_S(\underline{r}', \underline{\Omega}' \to \underline{r}, \underline{\Omega}) = \frac{\exp[-\tau(\underline{r}, \underline{r}', \underline{\Omega}')]}{\| \underline{r} - \underline{r}' \|^2} \sigma_s(\underline{r}, \underline{\Omega}' \to \underline{\Omega}) \delta\left(\underline{\Omega}' - \frac{\underline{r} - \underline{r}'}{\| \underline{r} - \underline{r}' \|} \right). \qquad (41)$$

Thus, the source function J satisfies the following integral equation

$$J(r, \Omega) = \int_{4\pi}\int_{V} \mathcal{K}_S(\underline{r}', \underline{\Omega}' \to \underline{r}, \underline{\Omega}) J(\underline{r}', \underline{\Omega}') d\underline{r}' d\underline{\Omega}' + q(\underline{r}, \underline{\Omega}). \qquad (42)$$

The intensity I can be expressed via J as $I = L_0^{-1}J$ where the operator L_0^{-1} transforms the function in accordance with Eq. (37). In many cases the solution of the integral equation (42) for the source function is a simpler task than Eq. (40) for the intensity since the integration over $\underline{\Omega}'$ can help to get rid of the angular variable. For example, in the case of isotopic source q, scattering ($\sigma_s = 1/4\pi$) and extinction (i.e., σ does not depend on $\underline{\Omega}$), the solution J becomes a function of the spatial variable while the corresponding intensity $I = L_0^{-1}J$ depends on both spatial and angular variables. The integral equation (42) is especially useful in the study of radiative transfer problems with simple forms of anisotropy. The integral equation for the intensity serves as a theoretical basis for many Monte Carlo models for radiative transfer process in various media.

9. Eigenvalues and Eigenvectors of the Radiative Transfer Equation

An *eigenvalue* of the radiative transfer equation is a number γ such that there exists a function $e(\underline{r}, \underline{\Omega})$ which satisfies

$$\gamma L_0 e = Se. \qquad (43)$$

Since the eigenvalue and *eigenvector* problem is formulated for zero boundary conditions (e=0), γ and $e(\underline{r}, \underline{\Omega})$ are independent on the incoming radiation. Under some general conditions [Vladimirov, 1963], the set of eigenvalues γ_k, k=0,1,2, … and eigenvectors $e_k(\underline{r}, \underline{\Omega})$, k=0,1,2, is a discrete set. The eigenvectors are mutually orthogonal, that is,

$$\int_{V}\int_{4\pi} \sigma(\underline{r}, \underline{\Omega}) e_k(\underline{r}, \underline{\Omega}) e_l(\underline{r}, \underline{\Omega}) d\underline{\Omega}\, d\underline{r} = \delta_{k,l}, \qquad (44)$$

where $\delta_{k,l}$ is the Kroneker symbol. The solution of the standard problem can be expanded in eigenvectors. The expansion in eigenvectors has mainly a theoretical value because the problem of finding these vectors is much more complicated than finding the solution of the transport equation. However, this approach can be useful to estimate integrals of the solution. Note that Eq. (43) is equivalent to finding of non-trivial solutions to the integral equation $\gamma e_k = L_0^{-1} S e_k$.

The discreteness of the eigenvalue set makes the radiative transfer problem similar to problems in linear algebra, i.e., the intensity can be represented as an infinite vector which satisfies an infinity number of linear algebraic equations given by a matrix determined by $L_0^{-1}S$.

The *transport equation* has a unique positive eigenvalue which corresponds to a unique positive [normalized in the sense of Eq. (44)] eigenvector. This eigenvalue is greater than the absolute magnitudes of the remaining eigenvalues. This means that only one eigenvector, say e_0, takes on positive values for any $\underline{r} \in V$ and $\underline{\Omega}$. This positive couplet of eigenvector and eigenvalue plays an important role in transport theory, for example, in neutron transport theory. The positive eigenvalue alone determines if the fissile assembly will function as a reactor, or as an explosive, or will melt. In vegetation canopy radiative transfer, the positive eigenvalue determines canopy absorption properties. The positive couplet, γ_0 and e_0, can be iterated based on the following property of the operator $T = L_0^{-1}S$

$$\gamma_0 = \lim_{m\to\infty} \gamma_{0,m},$$
$$e_0(\underline{r},\underline{\Omega}) = \lim_{m\to\infty} e_{0,m}(\underline{r},\underline{\Omega}). \tag{45}$$

Here

$$\gamma_{0,m+1} = \frac{\| T^{m+1}q \|_p}{\| T^m q \|_p},$$
$$e_{0,m}(\underline{r},\underline{\Omega}) = \frac{T^m q}{\| T^m q \|_p}, \tag{46}$$

where $\|\ldots\|_p$ is the norm defined by Eq. (33) and q is a source from the functional space H_p. The limits given by Eqs. (45) and (46) do not depend on p (i.e., on functional space H_p in which the problem was formulated) and the source $q \in H_p$ needed to initialize the sequences of $\gamma_{0,m}$ and $e_{0,m}$. If $p = 1$, value of $\gamma_{0,m+1}$ gives the probability that a photon from the source q scattered *m* times will be scattered again. The corresponding function $e_{0,m}(\underline{r},\underline{\Omega})$ is the probability density that a photon scattered *m* times will arrive at $\underline{r}$ along the direction $\underline{\Omega}$ without suffering a collision. These interpretations directly follow from the integral form Eq. (39) of the operator $T = L_0^{-1}S$ and the definition of the total interaction coefficient σ. Note that $\gamma_{0,m}$ and $e_{0,m}$ are related as

$$Te_{0,m} = \gamma_{0,m+1} e_{0,m+1}.$$

There is another formulation of the eigenvalues and eigenvectors in linear transport theory [Case and Zweifel, 1967]. Their approach is similar to that used in the theory of ordinary differential equations, i.e., solutions of the homogeneous problem ($q = 0$) are represented as the product of an exponential function of spatial variable and corresponding eigenfunction which depends on angular variable. Unlike the definition given by Eq. (33), the Case and Zweifel formulation results in both discrete and continuum of eigenvalues. The eigenfunctions corresponding to the continuum of the eigenvectors are Schwartz distributions, i.e., not functions in the usual sense. This approach allows for analytical solutions to the radiative transfer equation for a number of special cases and, therefore, provides in-depth understanding of the physics of radiative transfer process. For details of this approach, the reader is referred to Case and Zweifel [1967] and Bell and Glasstone [1970]. In this book, we follow the definition of the eigenvalue/eigenvector problem given by Eq. (43).

10. The Law of Energy Conservation

The stationary radiative transfer equation (24a) expresses the l*aw of energy conservation* for each spatial point r within V and for each direction $\underline{\Omega}$. The boundary condition (24b) describes energy exchange between V and the surrounding medium. Here we derive an expression of the energy conservation law for the domain V bounded by a surface δV, i.e., we perform integration of Eq. (24a) over V and the unit sphere 4π of directions,

$$\int_{4\pi\times V}\underline{\Omega}\bullet\nabla I\,d\underline{\Omega}\,d\underline{r}+\int_{4\pi\times V}[\sigma(\underline{r},\underline{\Omega})-\sigma_s'(\underline{r},\underline{\Omega})]I(\underline{r},\underline{\Omega})\,d\underline{\Omega}\,d\underline{r}=\int_{4\pi\times V}q(\underline{r},\underline{\Omega})\,d\underline{\Omega}\,d\underline{r}. \tag{47}$$

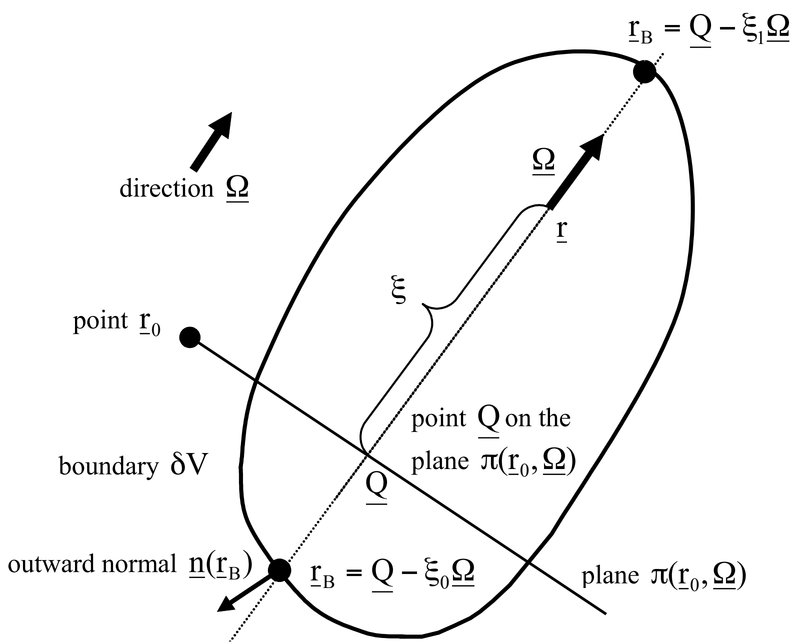

Figure 5. Representation of spatial points within a volume V bounded by the surface δV. Here $\pi(\underline{r}_0,\underline{\Omega})$ is a plane perpendicular to the direction $\underline{\Omega}$ and passing through a fixed point $\underline{r}_0$; ξ_1 and ξ_0 are distances between the point $\underline{Q}$ on the plane $\pi(\underline{r}_0,\underline{\Omega})$ and the boundary δV along the directions $\underline{\Omega}$ and $-\underline{\Omega}$, respectively; $\underline{r}_B$ denotes points on the boundary δV and $\underline{n}(\underline{r}_B)$ is the outward normal to δV at this point.

For a fixed direction $\underline{\Omega}$, let $\pi(\underline{r}_0,\underline{\Omega})$ be a plane perpendicular to $\underline{\Omega}$ and passing through a fixed point $\underline{r}_0$ (Fig. 5). Let $\underline{Q}$ be a variable point on the plane $\pi(\underline{r}_0,\underline{\Omega})$. Thus, the spatial point $\underline{r}$ within V can be represented as $\underline{r}=\underline{Q}+\xi\underline{\Omega}$. A volume element $d\underline{r}$ about $\underline{r}$ is $d\underline{r}=d\underline{Q}\,d\xi$, where $d\underline{Q}$ is a surface element on the plane $\pi(\underline{r}_0,\underline{\Omega})$ around the point $\underline{Q}$. It should be emphasized that the surface element $d\underline{Q}$ is perpendicular to the direction $\underline{\Omega}$. If one uses another elementary surface $d\underline{Q}'$ around the point $\underline{Q}$ which is perpendicular to a direction $\underline{\Omega}'$, the volume element $d\underline{r}$ is

$$d\underline{r} = \left|\underline{\Omega} \bullet \underline{\Omega}'\right| d\underline{Q}' d\xi. \tag{48}$$

Let ξ_1 and ξ_0 be distances between the point $\underline{Q}$ and the boundary δV along the directions $\underline{\Omega}$ and $-\underline{\Omega}$, respectively. For the first term in Eq. (47), we have

$$\begin{aligned}
\int_{4\pi\times V} \underline{\Omega} \bullet \nabla I \, d\underline{\Omega} \, d\underline{r} &= \int_{4\pi} d\underline{\Omega} \int_{\pi(\underline{r}_0,\underline{Q})} d\underline{Q} \int_{-\xi_0}^{\xi_1} \frac{dI(\underline{Q}+\xi\underline{\Omega},\underline{\Omega})}{d\xi} \\
&= \int_{4\pi} d\underline{\Omega} \int_{\pi(\underline{r}_0,\underline{Q})} d\underline{Q} \left[I(\underline{Q}+\xi_1\underline{\Omega},\underline{\Omega}) - I(\underline{Q}-\xi_0\underline{\Omega},\underline{\Omega}) \right] \\
&= \int_{4\pi} d\underline{\Omega} \int_{\pi(\underline{r}_0,\underline{Q})} d\underline{Q} \, I(\underline{Q}+\xi_1\underline{\Omega},\underline{\Omega}) - \int_{4\pi} d\underline{\Omega} \int_{\pi(\underline{r}_0,\underline{Q})} d\underline{Q} \, I(\underline{Q}-\xi_0\underline{\Omega},\underline{\Omega}) \\
&= \int_{\delta V} d\underline{r}_B \underbrace{\int_{\underline{n}(\underline{r}_B)\bullet\underline{\Omega}>0} d\underline{\Omega} \left|\underline{n}(\underline{r}_B) \bullet \underline{\Omega}\right| I(\underline{r}_B,\underline{\Omega})}_{\substack{\text{flux density of radiation leaving the medium} \\ \text{at } \underline{r}_B \text{ on the boundary}}} - \int_{\delta V} d\underline{r}_B \underbrace{\int_{\underline{n}(\underline{r}_B)\bullet\underline{\Omega}<0} d\underline{\Omega} \left|\underline{n}(\underline{r}_B) \bullet \underline{\Omega}\right| I(\underline{r}_B,\underline{\Omega})}_{\substack{\text{flux density of radiation entering the medium} \\ \text{through the point } \underline{r}_B \text{ on the boundary}}} \\
&= \underbrace{\int_{\delta V} F^{+}(\underline{r}_B) \, d\underline{r}_B}_{\substack{\text{energy of incoming} \\ \text{radiation}}} - \underbrace{\int_{\delta V} F^{-}(\underline{r}_B) \, d\underline{r}_B}_{\substack{\text{energy of outgoing} \\ \text{radiation}}} = E^{+}(\delta V) - E^{-}(\delta V).
\end{aligned} \tag{49}$$

In the second term of Eq. (47), the difference between σ and σ_s' is the absorption coefficient. This term, therefore, gives the amount of radiant energy at a wavelength λ absorbed by the domain V (in W m^{-1}). We use the symbol $E_a(V)$ to denote this variable. Finally, the right hand integral of Eq. (47) is the total amount of energy emitted by sources located within the volume V. We denote this quantity by q(V). Thus, the law of energy conservation for a given volume V bounded by a surface δV can be expressed as

$$E^{+}(\delta V) + E_a(V) = E^{-}(\delta V) + q(V), \tag{50}$$

that is, the amount of radiant energy reflected, $E^{+}(\delta V)$, and absorbed, $E_a(V)$, by the volume V is equal to the amount of energy, $E^{-}(\delta V)$, incident on the boundary δV and energy, q(V), emitted by the internal sources of the volume V.

11. Uniqueness Theorems

Here we formulate conditions under which the boundary value problem for the stationary radiative transfer problem has a unique solution. The radiative transfer problem is formulated for a domain V bounded by a reflecting surface δV. Photon interactins with the boundary are specified by Eq. (23).

The following parameters characterize optical properties of scatters and the entire medium as well as the interactions between the medium and the boundary.

The *maximum boundary reflectance*, $\rho_0(\delta V)$, quantifies the magnitude of boundary reflectance and is defined as

$$\rho_0(\delta V) = \sup_{\substack{\underline{r}'_B \in \delta V \\ \underline{n}(\underline{r}'_B)\bullet\underline{\Omega}'>0}} \frac{1}{\pi}\int_{\delta V} d\underline{r}_B \int_{\underline{n}(\underline{r}_B)\bullet\underline{\Omega}<0} \rho_B(\underline{r}'_B,\underline{\Omega}';\mathrm{r}_B,\underline{\Omega})\,|\,\underline{n}(\underline{r}_B)\bullet\underline{\Omega}\,|\,d\underline{\Omega}. \tag{51}$$

The *maximum optical path* is the maximum value of the optical distance between two points in the domain V [Eq. (38)]

$$\tau_0(V) = \sup_{\substack{\underline{r},\underline{r}'\in V \\ \underline{\Omega}\in 4\pi}} \tau(\underline{r},\underline{r}',\underline{\Omega}). \tag{52}$$

The *maximum single scattering albedo* is the maximum value of the single scattering albedo

$$\varpi_0(V) = \sup_{\underline{r}\in V,\,\underline{\Omega}\in 4\pi} \frac{\sigma'_s(r,\Omega)}{\sigma(r,\Omega)}. \tag{53}$$

The following theorem is a special case of Germogenova's maximum principle [Germogenova, 1986] which is proved here under the assumption of symmetry properties for the differential scattering coefficient, $\sigma_s(\underline{r},\underline{\Omega}'\to\underline{\Omega}) = \sigma_s(\underline{r},\underline{\Omega}\to\underline{\Omega}')$ and the boundary scattering function, $\rho_B(\underline{r}'_B,\underline{\Omega}';\underline{r}_B,\underline{\Omega}) = \rho_B(\underline{r}_B,-\underline{\Omega};\underline{r}'_B,-\underline{\Omega}')$. This restriction will be relaxed (cf. next section).

Theorem 1
Let $I(\underline{r},\underline{\Omega})$ *satisfies Eq. (24a) in the domain V and* $\varpi_0(V)\le 1$, $\tau_0(V)<\infty$ *and* $q=0$. *The following inequality holds true*

$$|\,I(\underline{r},\underline{\Omega})\,| \le \sup_{\underline{r}_B\in\delta V;\,\underline{\Omega}\bullet\underline{n}(\underline{r}_B)<0} \left| I(\underline{r}_B,\underline{\Omega})\right|, \tag{54}$$

for all $\underline{r}\in V+\delta V$ *and all directions.*

This theorem states that the intensity of radiation within V cannot exceed a maximum value of the intensity of radiation penetrating into V through the boundary δV. This theorem also presupposes that the incoming radiation field B is given by a bounded function. It means that this theorem cannot be applied if B contains a singular component, e.g., Dirac delta function. It is also assumed that the total interaction coefficient σ and the differential scattering coefficient σ_s are positive functions.

Proof

☐ Let $\bar{I} = \sup_{\underline{r}\in V+\delta V;\underline{\Omega}\in 4\pi} |I(\underline{r},\underline{\Omega})|$ where "supremum" is taken over all spatial points from V+δV and over all directions. We have

$$\begin{aligned}
\underline{\Omega} \bullet \nabla I(\underline{r},\underline{\Omega}) &= -\sigma(\underline{r},\underline{\Omega})\, I(\underline{r},\underline{\Omega}) + \int_{4\pi} d\underline{\Omega}'\,\sigma_s(\underline{r},\underline{\Omega}' \to \underline{\Omega})\, I(\underline{r},\underline{\Omega}) \\
&\le -\sigma(\underline{r},\underline{\Omega})\, I(\underline{r},\underline{\Omega}) + \int_{4\pi} d\underline{\Omega}'\,\sigma_s(\underline{r},\underline{\Omega}' \to \underline{\Omega}) \sup_{\underline{r}\in V;\underline{\Omega}'\in 4\pi} \left\{ I(\underline{r},\underline{\Omega}') \right\} \\
&= -\sigma(\underline{r},\underline{\Omega})\, I(\underline{r},\underline{\Omega}) + \bar{I}\,\frac{\sigma_s'(\underline{r},\underline{\Omega})}{\sigma(\underline{r},\underline{\Omega})}\,\sigma(\underline{r},\underline{\Omega}) \\
&\le [\varpi_0(V)\bar{I} - I(\underline{r},\underline{\Omega})]\sigma(\underline{r},\underline{\Omega}) \\
&\le [\bar{I} - I(\underline{r},\underline{\Omega})]\sigma(\underline{r},\underline{\Omega}). \qquad (55)
\end{aligned}$$

Note that the symmetry of the differential scattering coefficient was used to relate its integral over incident directions $\underline{\Omega}'$ to the scattering coefficient σ_s' (Section 2). Comparing the first and last term in (55), one obtains

$$[\bar{I} - I(\underline{r},\underline{\Omega})]\sigma(\underline{r},\underline{\Omega}) + \underline{\Omega} \bullet \nabla[\bar{I} - I(\underline{r},\underline{\Omega})] \ge 0. \qquad (56)$$

Multiplying this equation by $\exp[-\tau(\underline{r},\underline{r}-\xi\underline{\Omega},\underline{\Omega})]$ yields

$$-\frac{d}{d\xi}\left\{ \left[\bar{I} - I(\underline{r}-\xi\underline{\Omega},\underline{\Omega})\right] \exp\left(-\tau(\underline{r},\underline{r}-\xi\underline{\Omega},\underline{\Omega})\right) \right\} \ge 0.$$

Integrating the above over the interval [0,ξ] results in

$$\left[\bar{I} - I(\underline{r}-\xi\underline{\Omega},\underline{\Omega})\right] \exp\left[-\tau(\underline{r},\underline{r}-\xi\underline{\Omega},\underline{\Omega})\right] \le \bar{I} - I(\underline{r},\underline{\Omega}). \qquad (57)$$

Let us assume that the solution $I(\underline{r},\underline{\Omega})$ reaches its maximum at a point $\underline{r}_0$ within V and in a direction $\underline{\Omega}_0$, i.e., $\bar{I} = I(\underline{r}_0,\underline{\Omega}_0)$. Let ξ_B be the distance between the point $\underline{r}_0$ and the boundary δV along the direction ($-\underline{\Omega}_0$). It follows from (55) and $\tau_0(V) < \infty$ that

$$0 \le [\bar{I} - I(\underline{r}_0 - \xi_B\underline{\Omega}_0,\underline{\Omega}_0)]\exp\left(-\tau(\underline{r}_0,\underline{r}_0 - \xi_B\underline{\Omega}_0,\underline{\Omega}_0)\right) \le \bar{I} - I(\underline{r}_0,\underline{\Omega}_0) = 0,$$

which holds true if and only if $\bar{I} = I(\underline{r}_0 - \xi_B\underline{\Omega}_0,\underline{\Omega}_0)$. It means that the maximum of the solution $I(\underline{r},\underline{\Omega})$ taken over all internal points and over all directions cannot exceed the intensity of radiation entering the canopy in the direction $\underline{\Omega}_0$ through the point $\underline{r}_0$ on the boundary δV. This completes the proof.■

The inequality given by Eq. (57) for a more general case was originally derived by Germogenova [1986]. This results provides a theoretical justification to many existing radiation models. Based on Theorem 1, the following uniqueness theorem can be easily proved under the assumption of a symmetrical differential scattering coefficient σ_s and boundary bidirectional reflectance factor ρ.

Uniqueness Theorem
Let $\varpi \le 1$, $\rho_0 < 1$ and $\tau_0(V) < \infty$. The radiative regime within a given volume V of space bounded by a reflecting surface δV is uniquely determined by sources within V and the boundary conditions given by Eq. (23).

Proof
☐ Let $I_1(\underline{r},\underline{\Omega})$ and $I_2(\underline{r},\underline{\Omega})$ be two solutions of the transport equation (22) with boundary condition given by Eq. (23). The function $\psi(\underline{r},\underline{\Omega}) = I_1(\underline{r},\underline{\Omega}) - I_2(\underline{r},\underline{\Omega})$ satisfies Eq. (24a) with $q = 0$ and the boundary condition given by Eq. (23) with $q_B=0$. It follows from Theorem 1 and the symmetry $\rho_B(\underline{r}'_B,\underline{\Omega}';\underline{r}_B,\underline{\Omega}) = \rho_B(\underline{r}_B,-\underline{\Omega};\underline{r}'_B,-\underline{\Omega}')$ that the following inequality

$$
\begin{aligned}
|\psi(\underline{r},\underline{\Omega})| &\le \sup_{\substack{\underline{r}_B\in\delta V\\ \underline{\Omega}\bullet\underline{n}(\underline{r}_B)<0}} \left|B(\underline{r}_B,\underline{\Omega})\right| \\
&= \sup_{\substack{\underline{r}_B\in\delta V\\ \underline{\Omega}\bullet\underline{n}(\underline{r}_B)<0}} \left|\frac{1}{\pi}\int_{\delta V} d\underline{r}'_B \int_{\underline{\Omega}'\bullet\underline{n}(\underline{r}_B)>0} \rho_B(\underline{r}'_B,\underline{\Omega}';\underline{r}_B,\underline{\Omega})\,|\,\underline{n}(\underline{r}'_B)\bullet\underline{\Omega}'\,|\,\psi(\underline{r}'_B,\underline{\Omega}')\,d\underline{\Omega}'\right| \\
&\le \rho_0(\delta V) \sup_{\substack{\underline{r}_B\in\delta V\\ \underline{n}(\underline{r}_B)\bullet\underline{\Omega}<0}} |\psi(\underline{r}_B,\underline{\Omega})|,
\end{aligned}
\tag{58}
$$

is valid for all spatial points $\underline{r}\in V+\delta V$ and directions $\underline{\Omega}\in 4\pi$. Therefore,

$$
\sup_{\substack{\underline{r}_B\in\delta V\\ \underline{n}(\underline{r}_B)\bullet\underline{\Omega}<0}} |\psi(\underline{r}_B,\underline{\Omega})| \le \rho_0(V) \sup_{\substack{\underline{r}_B\in\delta V\\ \underline{n}(\underline{r}_B)\bullet\underline{\Omega}<0}} |\psi(\underline{r}_B,\underline{\Omega})|.
\tag{59}
$$

Since $\rho_0(V) < 1$, the inequality given by Eq. (59) holds true if and only if $|\psi(\underline{r},\underline{\Omega})| = 0$, i.e., $I_1(\underline{r},\underline{\Omega}) = I_2(\underline{r},\underline{\Omega})$. The uniqueness theorem is thus proved.■

12. General Case of Asymmetry

Theorem 1 and consequently the uniqueness theorem were proved under the assumption of certain symmetry in the differential scattering coefficient and the boundary bidirectional reflectance factor. This assumption was required to derive the inequalities given by Eqs. (55) and (58). To extend the validity of the uniqueness theorem to the general case, consider the adjoint formulation of the transport equation [Bell and Glasstone, 1970; Germogenova, 1986],

$$-\underline{\Omega}\bullet\nabla I^{*}(\underline{r},\underline{\Omega})+\sigma(\underline{r},\underline{\Omega})\,I^{*}(\underline{r},\underline{\Omega})=\int_{4\pi}\sigma_{s}(\underline{r},\underline{\Omega}\to\underline{\Omega}')\,I^{*}(\underline{r},\underline{\Omega}')\,d\underline{\Omega}', \tag{60}$$

$$I^{*}(\underline{r}_{B},\underline{\Omega})=B^{*}(\underline{r}_{B},\underline{\Omega}),\quad \underline{r}_{B}\in\delta V,\quad \underline{n}(\underline{r}_{B})\bullet\underline{\Omega}>0, \tag{61}$$

where

$$B^{*}(\underline{r}_{B},\underline{\Omega})=\frac{1}{\pi}\int_{\delta V}d\underline{r}'_{B}\int_{\underline{n}(\underline{r}_{B})\bullet\underline{\Omega}'<0}\rho_{B}(\underline{r}_{B},\underline{\Omega};\underline{r}'_{B},\underline{\Omega}')\,|\,\underline{n}(\underline{r}'_{B})\bullet\underline{\Omega}'\,|\,I^{*}(\underline{r}'_{B},\underline{\Omega}')\,d\underline{\Omega}'$$

$$+\,q^{*}(r_{B},\Omega),\quad \underline{n}(\underline{r}_{B})\bullet\underline{\Omega}>0. \tag{62}$$

The following differences should be noted between the standard formulation given by Eqs. (24) and (23) and its adjoint counterpart given by Eqs. (60)-(61): (a) the gradient operator $\underline{\Omega}\bullet\nabla$ has the opposite sign; (b) the incident $\underline{\Omega}'$ and scattering $\underline{\Omega}$ directions have been interchanged, i.e., $\underline{\Omega}'\to\underline{\Omega}$ in (23) and (24) becomes $\underline{\Omega}\to\underline{\Omega}'$ in Eqs. (60) and (62); and (c) the boundary condition (61) is formulated in terms of exiting photons, i.e., $\underline{n}(\underline{r}_{B})\bullet\underline{\Omega}>0$.

Physically, the adjoint radiative transfer problem describes the time-reversed photon flow. This gives us the hint that adjoint sources q^{*} describe the position of detectors while the adjoint transport equation describes the flow backward in time toward. Adjoint equations and their solutions play an important role in radiative transfer theory. Adjoint functions are, in a very real sense, orthogonal to the solutions of the radiative transfer equation [Bell and Glasstone, 1970; Germogenova, 1986]. For this and other reasons, they are widely used in perturbation theory and variational calculations relating to the behavior of 3D optical media. The properties of the solutions of the adjoint RTE are also used in the development of effective Monte Carlo calculations [Marchuk et al., 1980].

Consider the function $I_{0}^{*}(\underline{r},\underline{\Omega})=I^{*}(\underline{r},-\underline{\Omega})$. It satisfies the standard boundary value problem for the standard transport equation, i.e.,

$$\underline{\Omega}\bullet\nabla I_{0}^{*}(\underline{r},\underline{\Omega})+\sigma(\underline{r},-\underline{\Omega})I_{0}^{*}(\underline{r},\underline{\Omega})=\int_{4\pi}\sigma_{s}(\underline{r},-\underline{\Omega}\to-\underline{\Omega}')I_{0}^{*}(\underline{r},\underline{\Omega}')d\underline{\Omega}', \tag{63}$$

$$I_{0}^{*}(\underline{r}_{B},\underline{\Omega})=B^{*}(\underline{r}_{B},-\underline{\Omega}),\quad \underline{r}_{B}\in\delta V,\quad \underline{n}(\underline{r}_{B})\bullet\underline{\Omega}<0. \tag{64}$$

The uniqueness theorem can be applied to Eqs. (63)-(64) with the maximum boundary albedo, single scattering albedo and optical depth calculated using $\rho_{B}(\underline{r}_{B},-\underline{\Omega};\underline{r}'_{B},-\underline{\Omega}')$, $\sigma(\underline{r},-\underline{\Omega})$ and

$\sigma_s(\underline{r},-\underline{\Omega}\rightarrow-\underline{\Omega}')$. According to the Fredholm alternative [Bronshtein and Semendyaev, 1985, p. 783], a linear operator equation and its adjoint counterpart have a unique solution simultaneously. Therefore, we can use the adjoint transport equation to find the conditions under which it has a unique solution. The same conditions guarantee the uniqueness of the transport equation. Thus, the requirement for symmetry in the differential scattering coefficient and the boundary bidirectional reflectance factor can be relaxed.

Problem Sets

- **Problem 1**. The frequency of red light is $\nu=4.3\times10^{14}$ oscillations per second. What is a wavelength λ of red light?
- **Problem 2**. How are particle distribution functions in frequency and wavelength domains related?
- **Problem 3**. Let the differential solid angle $d\underline{\Omega}$ cuts an area consisting of points with polar and azimuthal angles from intervals $[\theta,\theta+d\theta]$ and $[\varphi,\varphi+d\varphi]$. Show that $d\underline{\Omega}=\sin\theta\, d\theta\, d\varphi$.
- **Problem 4**. How are the intensities in frequency and wavelength domains related?
- **Problem 5**. Some instruments (e.q., the LICOR quantum sensor) register broadband (i.e., integrated over a certain spectral interval) fluxes in mol m^{-2} s^{-1}. Therefore, it is often convenient to use the intensity $J(\underline{r},\nu,\underline{\Omega},t)$ expressed in mol m^{-3} s^{-1} sr^{-1} instead of $I(\underline{r},\nu,\underline{\Omega},t)$ in J m^{-2} sr^{-1}. How are intensities J, I and the particle distribution function f related?
- **Problem 6**. Let x, y and z be Cartesian coordinates of the point $\underline{r}_1$. Find Cartesian coordinates of the point $\underline{r}_2=\underline{r}_1+c\Delta t\underline{\Omega}$.
- **Problem 7**. Location $\underline{r}_1(t)$ of a photon at time t traveling along a direction $\underline{\Omega}$ can be expressed as $\underline{r}_1(t)=\underline{r}_B+ct\underline{\Omega}$ where $ct=\xi$ is the distance traversed by a photon in time interval t. Let x_B, y_B and z_B be Cartesian coordinates of the point $\underline{r}_B$. Find Cartesian coordinates of points $\underline{r}_1(t)$ and $\underline{r}_2(t)=\underline{r}_1+\Delta\xi\underline{\Omega}$ and their derivatives with respect to t.
- **Problem 8**. Show that if the extinction coefficient σ does not depend neither on spatial nor angular variables, $\tau(\underline{r}_1,\underline{r}_2,\underline{\Omega})=\sigma\|\underline{r}_1-\underline{r}_2\|$.
- **Problem 9**. Using Eq. (37) show that the volume Green's function for purely absorbing media (i.e., S=0) is given by
$$G_V(\underline{r}',\underline{\Omega}';\underline{r},\underline{\Omega})=\frac{\exp[-\tau(\underline{r},\underline{r}',\underline{\Omega}')]}{\|\underline{r}-\underline{r}'\|^2}\delta(\underline{\Omega}'\rightarrow\underline{\Omega})\delta\left(\underline{\Omega}-\frac{\underline{r}-\underline{r}'}{\|\underline{r}-\underline{r}'\|}\right).$$
- **Problem 10**. Let the total interaction coefficient σ be independent of the spatial and angular variables. Derive integral equations for the intensity and source function for isotropically scattering media with isotropic sources.
- **Problem 11**. Derive integral equations for the intensity and source function in plane geometry, i.e., a medium in which the total interaction coefficient, differential scattering coefficient and volume source are functions of the horizontal coordinate z.

- **Problem 12**. Let the total interaction coefficient σ be independent of the spatial and angular variables. Derive integral equations for the intensity and source function for a sphere with isotropic scattering and spherically symmetric volume sources. The volume source is said to by spherically symmetric if it depends on $\|\underline{r}\|$ and $\underline{\Omega} \bullet \underline{r}/\|\underline{r}\|$.
- **Problem 13**. Show that $Te_{0,m} = \gamma_{0,m+1} e_{0,m+1}$, where $\gamma_{0,m+1}$ and $e_{0,m}$ are eigenvalues and eigenvectors of transport equation, $\gamma L_0 e = Se$, and $T = L_0^{-1} S$ (cf. Section 9).
- **Problem 14**. Let δV be a reflecting boundary, i.e., a fraction of the medium leaving radiation can be reflected back into V. Assume that the boundary reflects as a Lambertian surface. The radiation I^- penetrating into V through δV is

$$I(\underline{r}_B, \underline{\Omega}) = \frac{\rho}{\pi} \int_{\underline{n}(\underline{r}_b) \bullet \underline{\Omega}' > 0} I(\underline{r}_B, \underline{\Omega}') |\underline{n}(\underline{r}_B) \bullet \underline{\Omega}'| d\underline{\Omega}' + q(\underline{r}_B, \underline{\Omega}), \quad \underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0 .$$

  Show that $(1-\rho)E^+(\delta V) + E_a(V) = E_q^-(\delta V)$, where

$$E_q^-(\delta V) = \int_{\delta V} d\underline{r}_B \int_{\underline{n}(\underline{r}_B) \bullet \underline{\Omega} < 0} d\underline{\Omega} \left| \underline{n}(\underline{r}_B) \bullet \underline{\Omega} \right| q(\underline{r}_B, \underline{\Omega}).$$

- **Problem 15**. Let V be the parallelepiped and δV_t, δV_b and δV_l are its top, bottom and lateral surfaces. Show that

$$E^+(\delta V) = E^+(\delta V_b) + E^+(\delta V_t) + E^+(\delta V_l),$$
$$E^-(\delta V) = E^-(\delta V_b) + E^-(\delta V_t) + E^-(\delta V_l).$$

- **Problem 16**. Let V be the parallelepiped and δV_t, δV_b and δV_l are its top, bottom and lateral surfaces. Write the energy conservation law in terms of canopy transmission, t, reflection, r, and horizontal energy flow, h, defined as

$$r = \int_{\underline{n}(\underline{r}_t) \bullet \underline{\Omega} > 0} I(\underline{r}_t, \underline{\Omega}') \left| \underline{n}(\underline{r}_t) \bullet \underline{\Omega}' \right| d\underline{\Omega}',$$

$$t = \int_{\underline{n}(\underline{r}_b) \bullet \underline{\Omega} > 0} I(\underline{r}_b, \underline{\Omega}') \left| \underline{n}(\underline{r}_b) \bullet \underline{\Omega}' \right| d\underline{\Omega}',$$

$$h = \int_{\underline{n}(\underline{r}_l) \bullet \underline{\Omega} > 0} I(\underline{r}_l, \underline{\Omega}') \left| \underline{n}(\underline{r}_l) \bullet \underline{\Omega}' \right| d\underline{\Omega}'.$$

- **Problem 17**. Prove the uniqueness theorem without assuming symmetrical differential scattering coefficient and boundary bidirectional reflectance factor.
- **Problem 18**. Prove that for the horizontally homogeneous media, solution of the transport problem depends on vertical coordinate only.

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