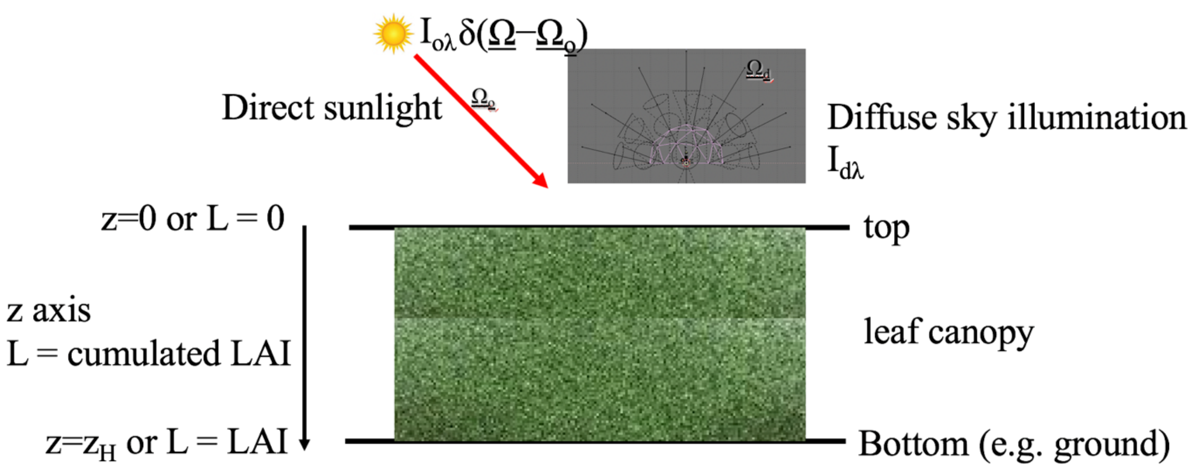
Chapter 4

Numerical Solution of the Radiative Transfer Equation

Myneni et al.



Chapter 4

Numerical Solution of the Radiative Transfer Equation

1. Transport Problem for Vegetation Canopies in One Spatial Dimension

We consider the one-dimensional radiative transfer equation for a leaf canopy confined between depths z = 0 at the top and $z = z_H$ at the bottom, that is the vertical ordinate is directed downwards. All directions are measured with respect to –z axis such that $\mu > 0$ for upward traveling directions. The canopy is assumed bounded at the bottom by a reflecting and absorbing ground and illuminated at the top by a mono-directional beam source (direct solar radiation) of intensity I_o along $\underline{\Omega}_o$ and a diffuse source (skylight) of intensity I_d, at wavelength λ. The appropriate transfer equation is

$$-\mu \frac{\partial}{\partial z} I_\lambda(z,\underline{\Omega}) + \sigma(z,\underline{\Omega}) I_\lambda(z,\underline{\Omega}) = \int_{4\pi} d\underline{\Omega}' \, \sigma_{s\lambda}(z,\underline{\Omega}' \to \underline{\Omega}) \, I_\lambda(z,\underline{\Omega}') \tag{1}$$

and the boundary conditions are

$$I_\lambda(z=0,\underline{\Omega}) = I_{o\lambda}\delta(\underline{\Omega}-\underline{\Omega}_o) + I_{d\lambda}(\underline{\Omega}), \quad \mu < 0, \tag{2a}$$

$$I_\lambda(z=z_H,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\underline{\Omega}' \rho_{s\lambda}(\underline{\Omega}' \to \underline{\Omega}) \,|\mu'|\, I_\lambda(z=z_H,\underline{\Omega}'), \quad \mu > 0. \tag{2b}$$

In the above, σ is the wavelength-independent total interaction cross section or the extinction coefficient, $\sigma_{s\lambda}$ is the wavelength-dependent differential scattering cross section and $\rho_{s\lambda}$ is the wavelength-dependent bi-directional reflectance function of the ground, or understory, beneath the vegetation canopy. The specific intensity I is thus wavelength dependent. However, for ease of expression, this dependence will be not explicitly shown for the reminder of this chapter. It is convenient to express the incident field as (cf. Chapter 4)

$$I_o(\underline{\Omega}) = \frac{f_{dir}}{|\mu_o|}\delta(\underline{\Omega}-\underline{\Omega}_o)F_{in}(z=0),$$
$$I_d(\underline{\Omega}) = (1-f_{dir})d_o(z=0,\underline{\Omega})F_{in}(z=0)$$

where f_{dir} is the fraction of total incident flux density at the top of the canopy, $F_{in}(z=0)$, is the total irradiance of the incident solar radiation at the top of canopy and d_o is the anisotropy of the diffuse source.

If the leaf normal orientation distribution function g_L is assumed independent of depth z in the canopy, the two cross sections in Eq. (1) can be written as (cf. Chapter 3)

$$\sigma(z,\underline{\Omega}') = u_L(z)G(\underline{\Omega}'), \tag{3a}$$
$$\sigma_S(z,\underline{\Omega}'\to\underline{\Omega}) = u_L(z)\frac{1}{\pi}\Gamma(\underline{\Omega}'\to\underline{\Omega}), \tag{3b}$$

where u_L is the leaf area density distribution, G is the geometry factor

$$G(\underline{\Omega}') = \frac{1}{2\pi}\int_{2\pi^+} d\underline{\Omega}_L g_L(\underline{\Omega}_L)|\underline{\Omega}_L \bullet \underline{\Omega}'|$$

and Γ is the area scattering phase function

$$\frac{1}{\pi}\Gamma(\underline{\Omega}'\to\underline{\Omega}) = \frac{1}{2\pi}\int_{2\pi^+} d\underline{\Omega}_L g_L(\underline{\Omega}_L)|\underline{\Omega}_L \bullet \underline{\Omega}'|\gamma_L(\underline{\Omega}_L;\underline{\Omega}'\to\underline{\Omega})$$

with γ_L being the leaf scattering phase function. The vertical coordinate z can be changed to cumulative leaf area index L by dividing Eq. (1) with u_L. The vegetation canopy is now contained between L = 0 at the top and L = L_H at the bottom, where L_H is the leaf area index of the canopy. The transport problem in one spatial dimension for a vegetation canopy illuminated at the top with unit flux density $[F_{in}(L=0)=1]$ and isotropic skylight is thus,

$$-\mu\frac{\partial}{\partial L}I(L,\underline{\Omega}) + G(L,\underline{\Omega})I(L,\underline{\Omega}) = \frac{1}{\pi}\int_{4\pi} d\underline{\Omega}'\Gamma(\underline{\Omega}'\to\underline{\Omega})I(L,\underline{\Omega}'), \tag{4a}$$
$$I(L=0,\underline{\Omega}) = \frac{f_{dir}}{|\mu_o|}\delta(\underline{\Omega}-\underline{\Omega}_o) + \frac{(1-f_{dir})}{\pi}, \quad \mu<0, \tag{4b}$$
$$I(L=L_H,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\underline{\Omega}'\rho_s(\underline{\Omega}'\to\underline{\Omega})|\mu'|I(L=L_H,\underline{\Omega}'), \quad \mu>0. \tag{4c}$$

2. Separation of Uncollided and Collided Intensities

It is convenient for numerical purposes and also to gain insight on the transport physics to separate the uncollided radiation field from the collided field, that is,

$$I(L,\underline{\Omega}) = I^0(L,\underline{\Omega}) + I^C(L,\underline{\Omega}) \tag{5}$$

where I^0 is the specific intensity of uncollided photons and I^C is the specific intensity of photons which experienced collisions with elements of the host medium. By introducing Eq. (5) in Eq. (4), the transport problem can be split into equations for the uncollided intensity,

$$-\mu\frac{\partial}{\partial L} I^0(L,\underline{\Omega}) + G(\underline{\Omega})\, I^0(L,\underline{\Omega}) = 0\,, \tag{6a}$$

$$I^0(L=0,\underline{\Omega}) = \frac{f_{dir}}{|\mu_o|}\delta(\underline{\Omega}-\underline{\Omega}_o) + \frac{(1-f_{dir})}{\pi}, \quad \mu<0, \tag{6b}$$

$$I^0(L=L_H,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\Omega'\, \rho_s(\underline{\Omega}\to\underline{\Omega})\left|\mu'\right| I^0(L=L_H,\underline{\Omega}')\,, \quad \mu>0, \tag{6c}$$

and collided intensity,

$$-\mu\frac{\partial}{\partial L} I^C(L,\underline{\Omega}) + G(\underline{\Omega})\, I^C(L,\underline{\Omega}) = Q(L,\underline{\Omega}) + S(L,\underline{\Omega})\,, \tag{7a}$$

$$I^C\left(L=0,\underline{\Omega}\right) = 0, \quad \mu<0\,, \tag{7b}$$

$$I^C(L=L_H,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\underline{\Omega}'\, \rho_s(\underline{\Omega}'\to\underline{\Omega})\left|\mu'\right| I^C(L=L_H,\underline{\Omega}')\,, \quad \mu>0, \tag{7c}$$

In the above, Q is the *first collision source*

$$Q(L,\underline{\Omega}) \equiv \frac{1}{\pi}\int_{4\pi} d\underline{\Omega}'\, \Gamma(\underline{\Omega}'\to\underline{\Omega})\, I^0(L,\underline{\Omega}') \tag{8a}$$

and S is the *distributed source*

$$S(L,\underline{\Omega}) = \frac{1}{\pi}\int_{4\pi} d\underline{\Omega}'\, \Gamma(\underline{\Omega}'\to\underline{\Omega})\, I^C(L,\underline{\Omega}')\,. \tag{8b}$$

3. Uncollided Problem

The solution to the uncollided problem [Eq. (6)] is,

$$I^0(L,\underline{\Omega}) = I^0(L=0,\underline{\Omega})\, P[\underline{\Omega},(L-0)], \quad \mu<0, \tag{9a}$$

$$I^0(L,\underline{\Omega}) = I^0(L = L_H,\underline{\Omega})\,P[\underline{\Omega},(L_H - L)], \quad \mu > 0 \tag{9b}$$

where

$$P[\underline{\Omega},(L2 - L1)] = \exp\left[-\frac{1}{|\mu|}G(\underline{\Omega})(L2 - L1)\right] \tag{9c}$$

denotes the probability of photons not experiencing collisions while traveling along $\underline{\Omega}$ between depth L1 and L2 (L2>L1). The downward uncollided intensity at the top of the canopy $I^0(L = 0,\underline{\Omega})$ in Eq. (9a) is given by the boundary condition, Eq. (6b). The upward intensity at the ground $I^0(L = L_H,\underline{\Omega})$ in Eq. (9b) can be evaluated as [cf. Eq. (6c)]

$$I^0(L = L_H,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\underline{\Omega}'\,\rho_s(\underline{\Omega} \to \underline{\Omega})\,|\mu'|\,I^0(L = 0,\underline{\Omega})\,P[\underline{\Omega}',(L_H - 0)], \quad \mu > 0. \tag{10}$$

If the ground reflectance is wavelength-independent, then the normalized uncollided radiation field is also wavelength-independent because the extinction coefficient is wavelength independent in vegetation canopies.

4. First Collision Problem

The collided problem specified by Eq. (7) is difficult to solve because of the distributed source term [Eq. (8b)]. Analytical solutions are possible only in the case of simple scattering kernels. As noted previously, the scattering phase function Γ is generally not rotationally-invariant, and this precludes the use of many standard techniques developed in transport theory. If scattering in the medium is weak, a single-scattering approximation may suffice. The corresponding transport problem

$$-\mu\frac{\partial}{\partial L}I^1(L,\underline{\Omega}) + G(\Omega)\,I^1(L,\underline{\Omega}) = Q(L,\underline{\Omega}), \tag{11a}$$

$$I^1(L = 0,\underline{\Omega}) = 0, \quad \mu < 0, \tag{11b}$$

$$I^1(L = L_H,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\underline{\Omega}'\,\rho_s(\underline{\Omega}' \to \underline{\Omega})\,|\mu'|\,I^1(L = L_H,\underline{\Omega}'), \quad \mu > 0 \tag{11c}$$

can be solved for the single-scattered intensity, I^1, that is, radiation intensity of photons scattered once, with just the first collision source term,

$$I^1(L,\underline{\Omega}) = \frac{1}{|\mu|}\int_0^L dL'\,Q(L',\underline{\Omega})\,P[\underline{\Omega},(L - L')], \quad \mu < 0, \tag{12a}$$

$$I^1(L,\underline{\Omega}) = \frac{1}{|\mu|}\int_{L}^{L_H} dL' Q(L',\underline{\Omega})\, P[\underline{\Omega},(L'-L)] + I^1(L = L_H,\underline{\Omega})P(\underline{\Omega}, L_H - L), \quad \mu > 0, \tag{12b}$$

where $I^1(L = L_H,\Omega)$ is given by Eq. (10).

5. Successive Orders of Scattering Approximation

The specific intensity of photons that experienced two collisions in the medium I^2 can be solved with knowledge of first scattered intensity I^1 as follows,

$$I^2(L,\underline{\Omega}) = \frac{1}{|\mu|}\int_{0}^{L} dL' S_1(L',\underline{\Omega})\, P[\underline{\Omega},(L - L')], \quad \mu < 0, \tag{13a}$$

$$I^2(L,\underline{\Omega}) = \frac{1}{|\mu|}\int_{L}^{L_H} dL' S_1(L',\underline{\Omega})\, P[\underline{\Omega},(L'-L)] + I^2(L = L_H,\underline{\Omega})P(\underline{\Omega}, L_H - L), \quad \mu > 0, \tag{13b}$$

where the distributed source term evaluated with first scattered intensity is

$$S_1(L,\underline{\Omega}) = \frac{1}{\pi}\int_{4\pi} d\underline{\Omega}'\,\Gamma(\underline{\Omega}' \to \underline{\Omega})\, I^1(L,\underline{\Omega}'). \tag{14}$$

The foregoing may be generalized for n-th order of scattering as

$$I^n(L,\underline{\Omega}) = \frac{1}{|\mu|}\int_{0}^{L} dL' S_{n-1}(L',\underline{\Omega})\, P[\underline{\Omega},(L - L')], \quad \mu < 0, \tag{15a}$$

$$I^n(L,\underline{\Omega}) = \frac{1}{|\mu|}\int_{L}^{L_H} dL' S_{n-1}(L',\underline{\Omega})\, P[\underline{\Omega},(L'-L)] + I^n(L = L_H,\underline{\Omega})P(\underline{\Omega}, L_H - L), \quad \mu > 0. \tag{15b}$$

The total intensity I and sources S can be evaluated as

$$I(L,\underline{\Omega}) = I^0(L,\underline{\Omega}) + \sum_{n=1}^{\infty} I^n(L,\underline{\Omega}), \tag{16a}$$

$$S(L,\underline{\Omega}) = \sum_{n=1}^{\infty} S_n(L,\underline{\Omega}).. \tag{16b}$$

In practice, the summation in Eq. (16) is limited to N-orders of scattering. Figure 1 shows convergence of

$$\varepsilon_n = S - \sum_{k=1}^{n} S_k,$$

at the top L=0, and the bottom, $L = L_H$, of the canopy. One can see that $\log \varepsilon_n$ follows a straight line if n exceeds a certain number, indicating that $\varepsilon_n \approx c\lambda^n$. Absolute value of the slope λ is the value of the convergence which depends on the re-collision probability, p, single scattering albedo, ϖ_0 and the maximum boundary reflectance, $\rho_0(\delta V)$. For the black soil problem (cf. Chapter 4), $\lambda = \varpi_0 \rho_0$. Thus the number of iterations needed to achieve a desired accuracy ε is inversely proportional to the rate of convergence, i.e. $n = |\ln \varepsilon / c| / \lambda$, and depends on canopy structure (p), leaf optics (ϖ_0)and boundary reflective properties ($\rho_0(\delta V)$). In general, λ is an increasing function of these variables and thus the higher p, ϖ_0, and $\rho_0(\delta V)$ are, the slower convergence is [Knjazikhin, 1990]. This method has been applied to model vegetation reflection by Myneni et al. [1987].

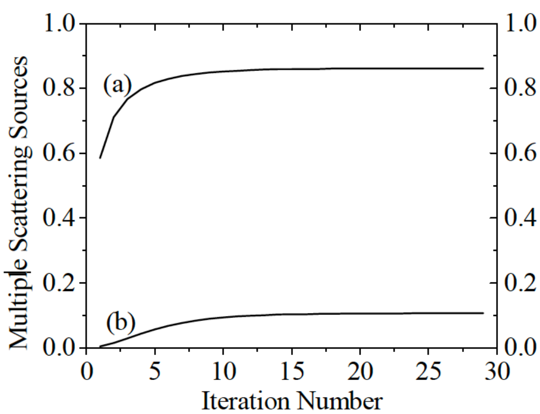


Figure 1. Convergence of the scattering sources S at the top of the canopy (a) and at the bottom of the canopy (b) during the numeric iteration process. The total Leaf Area Index (LAI) of the canopy is 5.0. The canopy is illuminated by diffuse incidence and by direct incidence from the direction (θ=150°, φ=0 °). The direct radiation accounts for 70% of the total incoming flux (F_{dir}=0.7). The leaf norm distribution of the canopy is planophile (mostly horizontal leaves). At the NIR wavelength ,the leaf reflectance ρ_{nir} is 0.475, and the leaf transmittance τ_{nir} is 0.45. The soil reflectance (ρ_{soil}) is 0.2. The threshold for error is considered 1e-05.

6. Gauss-Seidel Iteration Procedure for the Collided Problem

Consider the collided problem specified by Eq. (7). Discretize the angular variable $\underline{\Omega}$ into a finite number of directions, Ω_j, j = 1, 2, …, M, with the weights denoted by w_j. Similarly, discretize the spatial variable L, that is, divide L_H into N layers, each of thickness ΔL. We use the notation L_i and L_{i+1} to denote successive layers. The collided radiation at L_{i+2} and L_i in downward directions can be written as

$$I^C(L_{i+2},\Omega_j) = I^C(L_i,\Omega_j)*P(\Omega_j,2\Delta L) + 1/|\mu_j| * \int_{Li}^{Li+2} dL' \, J(L',\Omega_j)P[\Omega_j,(L_{i+2}-L')], \quad u < 0 , (17)$$

where J = Q + S is the source term, that is, the sum of first collision and distributed sources. If ΔL is small, then the following approximation is valid

$$\frac{1}{|\mu_j|}\int_{L_i}^{L_{i+2}} dL' J(L',\Omega_j) P[\Omega_j,(L_{i+2}-L')]$$

$$= \frac{1}{|\mu_j|}\int_{L_i}^{L_{i+2}} dL' J(L',\Omega_j) P[\Omega_j,(L'-L_i)]$$

$$\equiv J(L_{i+1},\Omega_j)\left[\frac{1}{G(\Omega_j)}\right]\left\{1-\exp\left[-\frac{1}{|\mu_j|}G(\Omega_j)2\Delta L\right]\right\} \quad (18)$$

where the source in the intervening layer is evaluated as

$$J(L_{i+1},\Omega_j) = \frac{1}{\pi}\sum_{k=1}^{M} w_k\, \Gamma(\Omega_k \to \Omega_j)\, I^C(L_{i+1},\Omega_k) + Q(L_{i+1},\Omega_j)\,. \quad (19)$$

A similar approximation and equation can be derived for upward directions (u > 0).

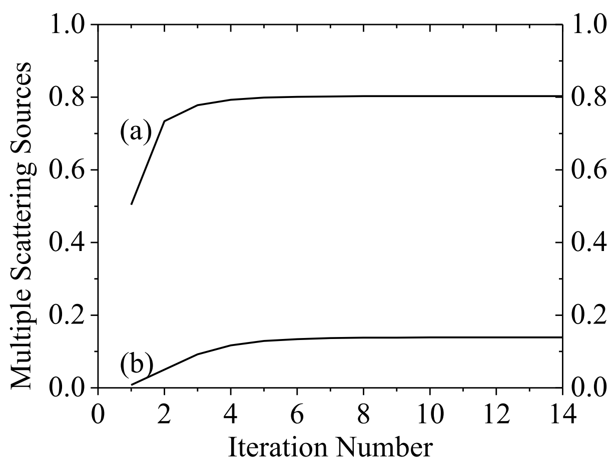


Figure 2. Convergence of multiple scattering sources (S) at the top of the canopy (a) and at the bottom of the canopy (b) during the iteration process. The total Leaf Area Index (LAI) of the canopy is 5. The canopy is illuminated by diffuse radiation and by direct solar radiation along the direction (θ=150°, φ=0°). Direct radiation accounts for 70% of the total incoming flux (F_{dir}=0.7). The leaf normal distribution of the canopy is planophile (mostly horizontal leaves). At NIR wavelengths, leaf reflectance (ρ_{nir}) is 0.475, leaf transmittance (τ_{nir}) is 0.45, and soil reflectance (ρ_{soil}) is 0.2.

The resulting system of linear algebraic equations can be solved iteratively for the unknowns $I^C(L_i,\Omega_j)$, i = 1, 2,…, N, j = 1, 2,…, M, using the Gauss-Seidel iteration procedure. The downward and upward intensities are computed from layer to layer for every iteration step. For instance, in the n-th iteration, the values of the downward intensities in layer i+2 are computed from the downward intensities of layer i and i+1 of the same iteration step and from the upward intensities of layer i+1 of the previous iteration step n-1. The convergence of the source with iteration count is shown in Fig. 2.

The advantages of this method are that the internal radiation field is readily available without additional labor and it is possible to account for vertical in-homogeneities in canopy structure and optics. The main limitation is that the iteration becomes tedious in optically deep canopies at strongly scattering wavelengths. This method has been used to model vegetation reflectance by Knyazikhin and Marshak [1991], and Liang and Strahler [1993].

7. Discrete Ordinates Method for the Collided Problem

We now consider numerical solution of the transport problem for the collided intensity in one spatial dimension using the discrete ordinates method as developed in neutron transport theory. In this method, photons are restricted to travel in a finite number of discrete directions, usually the quadrature directions, such that the angular integrals are evaluated with high precision. The spatial derivatives may be approximated by a finite difference scheme, to result in a set of equations which can be used to solve for the collided radiation field by iterating on the distributed source. This method has been used by Shultis and Myneni [1988] to model the radiation regime in vegetation canopies.

We consider the transport problem for the collided intensity [Eq. (7)]. The angular dependence of the transport equation is approximated by discretizing the angular variables μ and φ into a set of [N x M] discrete directions. The source terms are evaluated by numerical quadrature where $[\mu_i, \varphi_j]$ are the quadrature ordinates and the set of corresponding weights are $[w_i, \hat{w}_j]$. The transport equation for the collided intensity [Eq. (7)] can be written as

$$-\mu_i \frac{\partial}{\partial L} I^C(L,\Omega_{i,j}) + G(\Omega_{i,j})\, I^C(L,\Omega_{i,j}) = Q(L,\Omega_{i,j}) + S(L,\Omega_{i,j}) \qquad (20)$$

where the first collision and distributed sources are:

$$Q(L,\Omega_{i,j}) = \frac{1}{\pi}\sum_{n=1}^{N} w_n \sum_{m=1}^{M} \hat{w}_m \Gamma(\Omega_{n,m} \to \Omega_{i,j})\, I^o(L,\Omega_{n,m}), \qquad (21a)$$

$$S(L,\Omega_{i,j}) = \frac{1}{\pi}\sum_{n=1}^{N} w_n \sum_{m=1}^{M} \hat{w}_m \Gamma(\Omega_{n,m} \to \Omega_{i,j})\, I^C(L,\Omega_{n,m}). \qquad (21b)$$

The vegetation canopy contained between $L = 0$ and $L = L_H$ is divided into K layers of equal thickness ΔL. The spatial derivative in Eq. (20) is approximated as

$$\frac{\partial}{\partial L} I^C(L_{k+0.5},\Omega_{i,j}) = \frac{[I^C(L_{k+1},\Omega_{i,j}) - I^C(L_k,\Omega_{i,j})]}{\Delta L}, \qquad (22)$$

where k+0.5 is the center of the layer between the edges k and k+1. The discretized version of transport equation thus reads as

$$-\mu_i \frac{[I^C(L_{k+1},\Omega_{i,j}) - I^C(L_k,\Omega_{i,j})]}{\Delta L} + G(\Omega_{i,j})\, I^C(L_{k+0.5},\Omega_{i,j}) =$$

$$= Q(L_{k+0.5}, \Omega_{i,j}) + S(L_{k+0.5}, \Omega_{i,j}), \quad (23)$$

with k = 1, 2,…, K, i = 1, 2,…, N and j = 1, 2,…, M. To reduce the number of unknowns, a relation between cell-edge and cell-center collided intensities is required. Typically the following is used,

$$I^C(L_{k+0.5}, \Omega_{i,j}) \approx (1-\alpha) I^C(L_k, \Omega_{i,j}) + \alpha \times I^C(L_{k+1}, \Omega_{i,j}), \quad \mu < 0, \quad (24a)$$

$$I^C(L_{k+0.5}, \Omega_{i,j}) \approx (1-\alpha) I^C(L_{k+1}, \Omega_{i,j}) + \alpha \times I^C(L_k, \Omega_{i,j}), \quad \mu > 0 \quad (24b)$$

and if $\alpha = 0.5$, the cell-center intensity is the arithmetic average of the cell-edge intensities.

Equation (23) can be solved for $I^C(L_{k+1}, \Omega_{i,j})$ in terms of $I^c(L_k, \Omega_{i,j})$ in view of Eqs.(24) as

$$I^C(L_{k+1}, \Omega_{i,j}) = a_{i,j}\, I^C(L_k, \Omega_{i,j}) - b_{i,j}\, J(L_{k+0.5}, \Omega_{i,j}), \quad \mu < 0, \quad (25)$$

and for $I^C(L_k, \Omega_{i,j})$ in terms of $I^C(L_{k+1}, \Omega_{i,j})$ as

$$I^C(L_k, \Omega_{i,j}) = c_{i,j}\, I^C(L_{k+1}, \Omega_{i,j}) + d_{i,j}\, J(L_{k+0.5}, \Omega_{i,j}), \quad \mu > 0. \quad (26)$$

In the above,

$$a_{i,j} = \frac{1 + [1-\alpha] f_{i,j}}{[1 - \alpha f_{i,j}]}, \quad (27a)$$

$$b_{i,j} = \frac{f_{i,j}}{G(\Omega_{i,j})[1 - \alpha f_{i,j}]}, \quad (27b)$$

$$c_{i,j} = \frac{1 - [1-\alpha] f_{i,j}}{[1 + \alpha f_{i,j}]}, \quad (27c)$$

$$d_{i,j} = \frac{f_{i,j}}{G(\Omega_{i,j})[1 + \alpha f_{i,j}]}, \quad (27d)$$

$$f_{i,j} = \frac{G(\Omega_{i,j}) \Delta L}{\mu_i}, \quad (27e)$$

$$J(L_{k+0.5}, \Omega_{i,j}) = Q(L_{k+0.5}, \Omega_{i,j}) + S(L_{k+0.5}, \Omega_{i,j}). \quad (27f)$$

These equations are of the standard form except for the angular dependence of the coefficients $a_{i,j}$ through $d_{i,j}$ because of the geometry factor G. The set of Eqs. (25) and (26) can be used to solve for collided intensity as follows. While sweeping downwards in the phase space, Eq. (25) is used to step successively down in the mesh. At the bottom, the boundary condition is handled as

$$I^C(L_{K+1}, \Omega_{i,j}) = \frac{1}{\pi} \sum_{n=1}^{N/2} w_n \sum_{m=1}^{M} \hat{w}_m\, \rho_s(\Omega_{n,m} \to \Omega_{i,j}) \left|\mu_n\right| I^C(L_{K+1}, \Omega_{n,m}). \quad (28)$$

Note that i = (N/2)+1,…, N in the above. Now, Eq. (26) is used to sweep through the grid in upward directions. The distributed source is upgraded [Eq. (21b)] using the relations between cell-edge and cell-center intensities [Eqs. (24)]. This procedure is repeated until the cell-edge intensities in successive iterations do not differ by more than a preset threshold value – the convergence of the source with iteration is shown in Fig. 3.

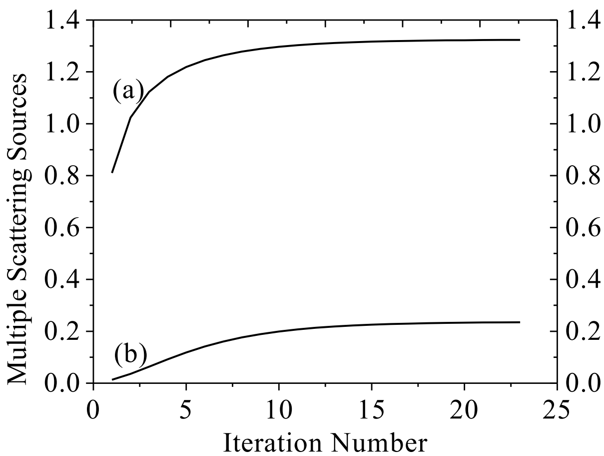
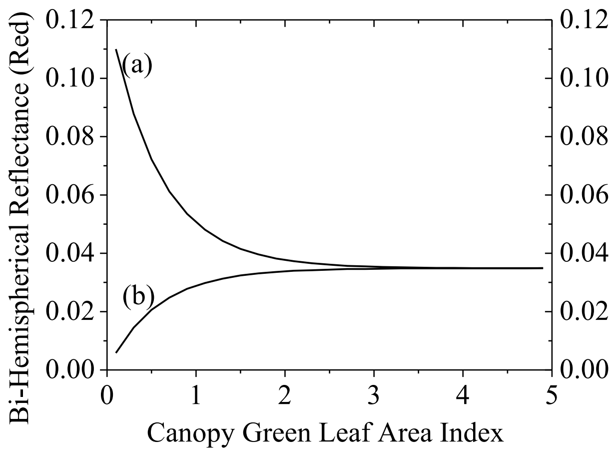
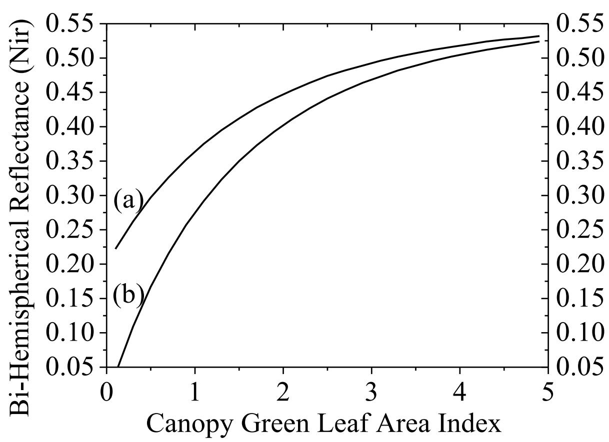


Figure 3. Convergence of multiple scattering sources (S) at the top of the canopy (a) and at the bottom of the canopy (b) during the iteration process. The total Leaf Area Index (LAI) of the canopy is 5. The canopy is illu-minated by diffuse radiation, and by direct solar radia-tion along the direction (θ=150°, φ=0 °). Direct radia-tion accounts for 70% of the total incoming flux (f_{dir}=0.7). The leaf normal distribution of the canopy is planophile (mostly horizontal leaves). At NIR wavelengths, leaf reflectance (ρ_{nir}) is 0.475, leaf trans-mittance (τ_{nir}) is 0.45, and soil reflectance (ρ_{soil}) is 0.2.

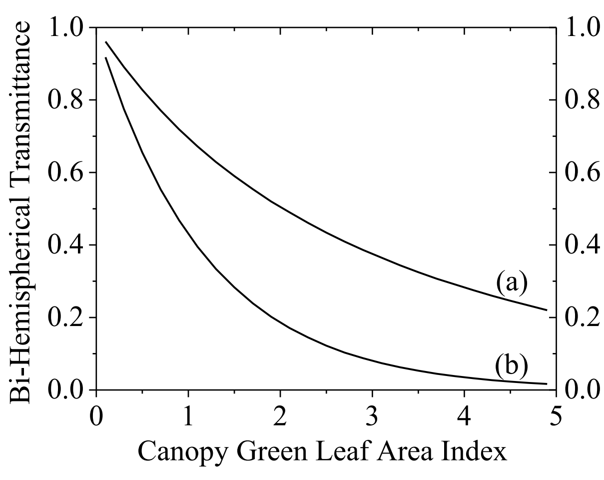
Numerical results illustrating how vegetation canopy reflectance (and transmittance) changes with respect to leaf area index and sun-view inclination angles are shown in Fig. 4.



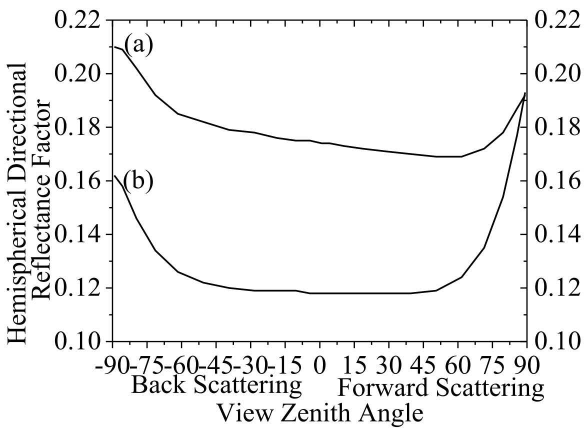
Bi-Hemispherical Reflectance (BHR) at red wavelengths (~ 0.7μm) as a function of canopy green leaf area index (LAI). Ground reflectance (ρ_{soil}) for the two different soil types is (a) 0.125 (typical soil reflectance); and (b) 0 (black soil). The other problem parameters are as in the standard problem described in the figure caption.



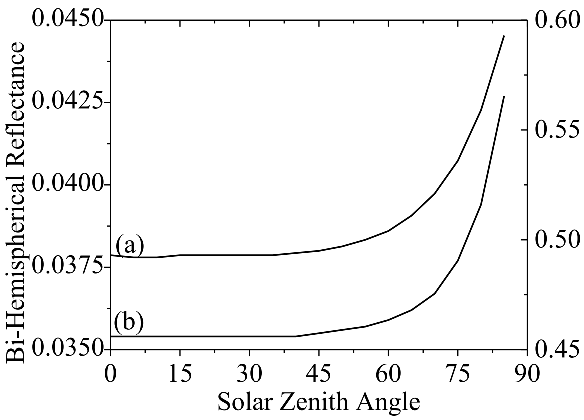
Bi-Hemispherical Reflectance (BHR) at NIR wavelengths (~0.9μm) as a function of canopy green leaf area index (LAI). Ground reflectance (ρ_{soil}) for the two different soil types is: (a) 0.2 (typical soil reflectance); and (b) 0 (black soil). The other problem parameters are as in the standard problem described in the figure caption.



Bi-Hemispherical Transmittance (BHT) at red (~0.7μm) and NIR (~0.9μm) wavelengths as a function of canopy green leaf area index (LAI). The other problem parameters are as in the standard problem described in the figure caption.



Hemispherical Directional Reflectance Factor (HDRF) in the solar principal plane. For back scattering, $\varphi_{view} = \varphi_0 + \pi$, and for forward scattering, $\varphi_{view} = \varphi_0$. The total Leaf Area Index (LAI) of the canopy is 3.0, and the soil reflectance is 0.2. Optical parameters of the canopy for the two lines are: (a) leaf reflectance is 0.7, and leaf transmittance is 0.225; (b) leaf reflectance is 0.225, and leaf transmittance is 0.7. The other problem parameters are as in the standard problem described in the figure caption.



Bi-Hemispherical Reflectance (BHR) at NIR (right axis) and RED (left axis) wavelengths as a function of solar zenith angle. The total Leaf Area Index (LAI) of the canopy is 3.0. The other problem parameters are as in the standard problem described in the figure caption.

Figure 4. The illumination conditions for these cases are as follows – the canopy is illuminated by diffuse solar radiation and direct solar radiation along the direction (θ=150°, φ=0 °). Direct solar radiation accounts for 70% of the total incoming flux (F_{dir}=0.7). The leaf normal distribution is planophile (mostly horizontal leaves). At NIR wavelengths (a), leaf reflectance (ρ_{nir}) is 0.475 and leaf transmittance (τ_{nir}) is 0.45. At red wavelengths (b), (ρ_{red}) is 0.075, and (τ_{red}) is 0.035.

8. Two-Stream Approximations

In cases where the angular distribution of the radiation field is of less interest, the transport equation can be angle-integrated to derive the appropriate equations for radiation fluxes. One example is the case where one is interested in the evaluation of hemispherical reflectances, BHR or DHR. The resulting differential equations can be solved analytically in some cases. Methods based on flux approximations have been widely used to model vegetation canopy radiation regime because of their simplicity and the possibility of analytical solutions, ([Allen and Richardson, 1968]; [Suits, 1972]; [Dickinson, 1983]; [Verhoef, 1984]; [Sellers, 1985]; amongst others).

Consider the transport problem stated by Eq. (1) and (2). The downward flux density F^d is defined as

$$F^d(L) = \int_{2\pi^-} d\underline{\Omega}\, |\mu|\, I(L,\underline{\Omega}) .$$

Integrating Eq. (1) over all downward directions, but with change of vertical coordinate z to cumulative leaf area index L,

$$\frac{\partial}{\partial L} F^d(L) + F^d(L) \frac{\int_{2\pi-} d\underline{\Omega}\, G(\underline{\Omega})\, I(L,\underline{\Omega})}{F^d(L)}$$
$$= F^u(L) \frac{\int_{2\pi-} d\underline{\Omega} \int_{2\pi+} d\underline{\Omega}' \frac{1}{\pi} \Gamma(\underline{\Omega}' \to \underline{\Omega})\, I(L,\underline{\Omega}')}{F^u(L)}$$
$$+ F^d(L) \frac{\int_{2\pi-} d\underline{\Omega} \int_{2\pi-} d\underline{\Omega}' \frac{1}{\pi} \Gamma(\underline{\Omega}' \to \underline{\Omega})\, I(L,\underline{\Omega}')}{F^d(L)} , \quad (29)$$

and simplifying results in a differential equation for the downward flux density

$$\frac{\partial}{\partial L} F^d(L) + K_1^d F^d(L) = K_2^d F^u(L) + K_3^d F^d(L). \quad (30)$$

Similarly, a differential equation for the upwards flux density can be derived

$$-\frac{\partial}{\partial L} F^u(L) + K_1^u F^u(L) = K_2^u F^u(L) + K_3^d F^d(L). \quad (31)$$

The initial values for Eqs. (30) and (31) are

$$F^d(L=0) = f_{dir} |\mu_0| I_0 + (1 - f_{dir}) \int_{2\pi-} d\underline{\Omega}_d |\mu_d| I_d(\underline{\Omega}_d), \quad (32)$$

$$F^u(L = L_H) = r_s F^d(L = L_H) \quad (33)$$

where r_s is the hemispherical reflectance of the ground underneath the canopy. While these initial value problems seem simple enough, it is not easy to rigorously derive expressions for the coefficients K in the general case of distributed leaf normals and anisotropic scattering kernels. However, approximate expressions generally surface in many practical instances.

We consider the simple case of a horizontally homogeneous leaf canopy consisting of horizontal leaves. The geometry factor $G(\underline{\Omega}) = |\mu|$ and the area scattering phase function is simply

$$\Gamma(\underline{\Omega}' \to \underline{\Omega}) = \begin{cases} \tau_L \mu\mu', & \mu\mu' > 0, \\ \rho_L |\mu\mu'|, & \mu\mu' < 0. \end{cases}$$

Specular reflection from leaf surfaces is ignored in this formulation. In the above ρ_L and τ_L are the leaf hemispherical reflectance and transmittance. The coefficients for this canopy are, $K_1^d = 1$, $K_2^d = \rho_L$, $K_3^d = \tau_L$, $K_1^u = 1$, $K_2^u = \tau_L$ and $K_3^u = \rho_L$. The differential equations (30) and (31) can be rewritten as

$$\frac{\partial}{\partial L} F^d(L) = \rho_L F^u(L) + (\tau_L - 1) F^d(L), \tag{34a}$$

$$-\frac{\partial}{\partial L} F^u(L) = \rho_L F^d(L) + (\tau_L - 1) F^u(L). \tag{34b}$$

That is, the changes in the downward flux density are given by the sum of backscattered upward flux density (the gain term) and the fraction of downward flux density that is not forward scattered (the loss term). Similarly, the changes in the upward flux density are given by the sum of backscattered downward flux density (the gain term) and the fraction of upward flux density that is not forward scattered (the loss term). The transport equations can be solved with the initial values [Eqs. (32) and (33)] to obtain downward and upward radiation flux density in the medium.

If the leaves are spherically distributed, the geometry factor $G(\underline{\Omega}) = 0.5$ and the area scattering phase function is given by

$$\Gamma(\underline{\Omega}' \to \underline{\Omega}) = \frac{w_L}{3\pi}(\sin\beta - \beta\cos\beta) + \frac{\tau_L}{\pi}\cos\beta,$$

where $\beta = \arccos(\underline{\Omega}' \bullet \underline{\Omega})$, again on the assumption of negligible specular reflection from leaf surfaces. Analytical expressions for the coefficients K and solution of the corresponding transport equations for downward and upwards fluxes are straightforward, although tedious.

9. The Hot-Spot Effect

The hot spot results from considerations of the relative sizes of scatterers in the canopy (leaves, branches, twigs, etc.) in relation to the wavelength of the radiation. Shadowing is ubiquitous and mutual shadowing of scatterers is predominant. The reflected radiation field tends to peak about the retro-illumination direction under such cases - this is termed the hot spot effect in vegetation remote sensing (Fig. 5). The shape and magnitude of the hot spot depends on the structure of the medium and is especially pronounced at shorter wavelengths where scattering is weak because the shadows are darker. The hot spot phenomenon is observational evidence of the limitations of theoretical developments that ignore scatterer size and resulting directional correlations in the interaction cross sections. Inclusion of such considerations in the transport equation is feasible but complicated [Myneni et al., 1991]. Here we present a simple methodology for inclusion of the hot spot phenomenon in the transport equation. A model of the hot spot effect in the limit of single scattering can be found in Kuusk [1985].

The hot spot effect can be included in the transport equation through the use of a modified total interaction cross section $\tilde{\sigma}$ (cf. Marshak [1989]),

$$\tilde{\sigma}(L,\underline{\Omega},\underline{\Omega}') = \begin{cases} \sigma(L,\underline{\Omega})\ \{1-\exp[-\kappa\, D(\underline{\Omega}\bullet\underline{\Omega}')]\}, & (\underline{\Omega}\bullet\underline{\Omega}')<0, \\ \sigma(L,\underline{\Omega}), & (\underline{\Omega}\bullet\underline{\Omega}')>0, \end{cases} \tag{35}$$

where κ is a parameter related to the ratio of vegetation height to characteristic leaf dimension. Its values were estimated to be between 1 and 8 from experimental data. The distance D is given by

$$D(\underline{\Omega},\underline{\Omega}') = \sqrt{\frac{1}{\mu'^2}+\frac{1}{\mu^2}+\frac{2(\underline{\Omega}\bullet\underline{\Omega}')}{|\mu'\mu|}}\ .$$

This particular model for the modified total interaction cross section has two desirable features, namely, that for $\underline{\Omega} = -\underline{\Omega}'$, $\tilde{\sigma}$ vanishes to result in the hot spot, and for large scattering angles, it approaches the standard cross section σ. Note that $\tilde{\sigma}$ is always positive.

Figure 5. The hot spot effect of a vegetation canopy.

Consider the one-dimensional leaf canopy transport problem. Let the total radiation intensity be represented as $I = I^0+I^1+I^m$, that is, as the sum of uncollided, first collision and multiple collision intensities. Further, assume for ease of presentation that the incident diffuse skylight I_d=0, that is, f_{dir}=1. The transport problems for I^0 and I^1, specified by Eqs. (6) and (11), are modified using $\tilde{\sigma}$ instead of σ, or equivalently, instead of $\tilde{G}$ instead of G [cf. Eq. (3a)]. The solutions for the downward intensities I^0 and I^1 given by Eqs. (9a) and (12a) remain unchanged since $\widetilde{G} = G$ for $(\underline{\Omega} \bullet \underline{\Omega}') > 0$. The upward uncollided and first collided radiation intensities are, however, modified because of the modified cross section. They read

$$I^0(L,\underline{\Omega}) = I^0(L = L_H,\underline{\Omega})\, P[\underline{\Omega},\underline{\Omega}_o,(L_H - L)], \quad \mu > 0, \tag{36a}$$

$$I^1(L,\underline{\Omega}) = \frac{1}{|\mu|}\int_L^{L_H} dL'\, Q(L',\underline{\Omega})\, P[\underline{\Omega},\underline{\Omega}_o,(L' - L)], \quad \mu > 0 \tag{36b}$$

where

$$P[\underline{\Omega},\underline{\Omega}_o,(L2 - L1)] = \exp\left[-\frac{1}{|\mu|}\widetilde{G}(\underline{\Omega},\underline{\Omega}_o)(L2 - L1)\right] \tag{37}$$

denotes the probability of photons not experiencing collisions while traveling along $\underline{\Omega}_o$ from the top of the canopy (L = 0) to depth L2 and along $\underline{\Omega}$ from L2 and L1 (L2 > L1). This is the required bi-directional gap probability for implementing the hot spot effect.

The multiple collision transport problem is similar to the collided intensity transport problem specified by Eqs. (7) except that the first collision source Q in Eq. (7a) is replaced by the second collision source,

$$Q(L,\underline{\Omega}) = \frac{1}{\pi}\int_{4\pi} d\underline{\Omega}'\, \Gamma(\underline{\Omega}' \to \underline{\Omega})\, I^1(L,\underline{\Omega}').$$

The above formulation allows simulation of the hot spot effect with transport equations. This is accomplished in an *ad hoc* manner by utilizing a modified interaction cross section for the uncollided and first collided intensities arising from incident solar radiation and using the standard cross section for multiply collided transport problem. The uncollided and collided intensities due to incident diffuse skylight can also be solved the standard way utilizing the unmodified cross section.

While the above formalism allows inclusion of the hot spot effect, it does result in a system that violates the energy conservation principle, because the transport problem for the collided intensity is, strictly speaking,

$$-\mu\frac{\partial}{\partial L} I^C(L,\underline{\Omega}) + G(L,\underline{\Omega}) I^C(L,\underline{\Omega}) = Q(L,\underline{\Omega}) + S(L,\underline{\Omega}) + [G(\underline{\Omega}) - \widetilde{G}(\underline{\Omega},\underline{\Omega}_o)] I^C(L,\underline{\Omega}),$$

$$I^C(L = 0,\underline{\Omega}) = 0, \quad \mu < 0,$$

$$I^1(L = L_H,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\underline{\Omega}'\, \rho_s(\underline{\Omega}' \to \underline{\Omega})\, |\mu'|\, I^C(L = L_H,\underline{\Omega}'), \quad \mu > 0.$$

This is equivalent to a transport equation for the total intensity I [Eqs. (1)] with an additional fictitious internal source, $[G(\underline{\Omega}) - \widetilde{G}(\underline{\Omega},\underline{\Omega}_o)]\cdot I^C(L,\underline{\Omega})$, which results in violation of the energy conservation principle. This has implications for the inverse problems where this principle is used as a constraint.

10. Discrete Ordinates Method in Three Spatial Dimensions

We consider a spatially heterogeneous leaf canopy contained between $0 < z < Z_s$, $0 < x < X_s$, $0 < y < Y_s$, where X_s, Y_s, and Z_s denote the dimensions of the stand. The canopy is assumed homogeneously illuminated on the top and lateral faces by a mono-directional beam source (direct solar radiation) of intensity I_o along $\underline{\Omega}_o$ and a diffuse source (skylight) of intensity I_d. The ground below the canopy is assumed to reflect and absorb the radiation field non-homogeneously. The radiation intensity in the governing transport equation,

$$-\mu\frac{\partial}{\partial z}I(\underline{r},\underline{\Omega}) + \eta\frac{\partial}{\partial y}I(\underline{r},\underline{\Omega}) + \xi\frac{\partial}{\partial y}I(\underline{r},\underline{\Omega}) + \sigma(\underline{r},\underline{\Omega})I(\underline{r},\underline{\Omega}) = \int_{4\pi} d\underline{\Omega}'\sigma_s(\underline{r},\underline{\Omega}'\rightarrow\underline{\Omega})I(\underline{r},\underline{\Omega}'),$$

is separated into uncollided (I^0) and collided (I^C) fields. In the above, $\underline{r}\equiv(x,y,z)$, and μ, η and ξ are the direction cosines with respect to the z, y and x dimensions.

The uncollided problem is given by the transport equation

$$-\mu\frac{\partial}{\partial z}I^0(\underline{r},\underline{\Omega}) + \eta\frac{\partial}{\partial y}I^0(\underline{r},\underline{\Omega}) + \xi\frac{\partial}{\partial y}I^0(\underline{r},\underline{\Omega}) + \sigma(\underline{r},\underline{\Omega})I^0(\underline{r},\underline{\Omega}) = 0, \quad (38a)$$

and the boundary conditions

$$I^0(x,y,z=0,\underline{\Omega}) = I_o\delta(\underline{\Omega}-\underline{\Omega}_o) + I_d(\underline{\Omega}), \quad \mu<0, \quad (38b)$$
$$I^0(x=0,y,z,\underline{\Omega}) = I_o\delta(\underline{\Omega}-\underline{\Omega}_o) + I_d(\underline{\Omega}), \quad \xi>0 \text{ and } \mu<0, \quad (38c)$$
$$I^0(x=X_s,y,z,\underline{\Omega}) = I_o\delta(\underline{\Omega}-\underline{\Omega}_o) + I_d(\underline{\Omega}), \quad \xi<0 \text{ and } \mu<0, \quad (38d)$$
$$I^0(x,y=0,z,\underline{\Omega}) = I_o\delta(\underline{\Omega}-\underline{\Omega}_o) + I_d(\underline{\Omega}), \quad \eta>0 \text{ and } \mu<0, \quad (38e)$$
$$I^0(x,y=Y_s,z,\underline{\Omega}) = I_o\delta(\underline{\Omega}-\underline{\Omega}_o) + I_d(\underline{\Omega}), \quad \eta<0 \text{ and } \mu<0, \quad (38f)$$
$$I^0(x,y,z=Z_s,\underline{\Omega}) = \frac{1}{\pi}\int_{2\pi^-} d\underline{\Omega}'\rho_s(x,y,\underline{\Omega}'\rightarrow\underline{\Omega})\,|\mu'|\,I^0(x,y,z=Z_s,\underline{\Omega}'), \quad \mu>0. \quad (38g)$$

In the above, ρ_s is the wavelength-dependent bi-directional reflectance function of the ground below the canopy. The solution to the uncollided problem is

$$I^0(\underline{r},\underline{\Omega}) = I^0(\underline{r}_B,\underline{\Omega})\,P[\underline{\Omega},|\underline{r}-\underline{r}_B|], \quad \mu<0, \quad (39a)$$

$$I^0(\underline{r},\underline{\Omega}) = I^0(\underline{r}_H,\underline{\Omega})\, P[\underline{\Omega}, |\underline{r}-\underline{r}_H|], \quad \mu > 0 \tag{39b}$$

where r_B is a point on the top or the lateral faces of the canopy and r_H is a point on the ground below the canopy. The quantity

$$P[\underline{\Omega}, |\underline{r}_2 - \underline{r}_1|] = \exp\left[- \int_0^{|\underline{r}_2-\underline{r}_1|} ds\, \sigma(\underline{r}_1 + s\underline{\Omega}, \underline{\Omega}) \right] \tag{40}$$

denotes the probability of photons not experiencing collisions while traveling along $\underline{\Omega}$ between the points $\underline{r}_1$ and $\underline{r}_2$.

The collided problem is given by the transport equation

$$-\mu \frac{\partial}{\partial z} I^C(\underline{r},\underline{\Omega}) + \eta \frac{\partial}{\partial y} I^C(\underline{r},\underline{\Omega}) + \xi \frac{\partial}{\partial y} I^C(\underline{r},\underline{\Omega}) + \sigma(\underline{r},\underline{\Omega})\, I^C(\underline{r},\underline{\Omega}) = S(\underline{r},\underline{\Omega}) + Q(\underline{r},\underline{\Omega}), \tag{41a}$$

where the distributed source S and first collision source Q are

$$S(\underline{r},\underline{\Omega}) = \int_{4\pi} d\underline{\Omega}' \sigma_s(\underline{r};\underline{\Omega}' \to \underline{\Omega})\, I^C(\underline{r},\underline{\Omega}'),$$
$$Q(\underline{r},\underline{\Omega}) = \int_{4\pi} d\underline{\Omega}' \sigma_s(\underline{r};\underline{\Omega}' \to \underline{\Omega})\, I^0(\underline{r},\underline{\Omega}').$$

The boundary conditions for the collided intensity are,

$$I^C(x,y,z=0,\underline{\Omega}) = 0, \quad \mu < 0, \tag{41b}$$
$$I^C(x=0,y,z,\underline{\Omega}) = 0, \quad \xi > 0, \tag{41c}$$
$$I^C(x=X_s,y,z,\underline{\Omega}) = 0, \quad \xi < 0, \tag{41d}$$
$$I^C(x,y=0,z,\underline{\Omega}) = 0, \quad \eta > 0, \tag{41e}$$
$$I^C(x,y=Y_s,z,\underline{\Omega}) = 0, \quad \eta < 0, \tag{41f}$$
$$I^C(x,y,z=Z_s,\underline{\Omega}) = \frac{1}{\pi} \int_{2\pi^-} d\underline{\Omega}' \rho_s(x,y,\underline{\Omega}' \to \underline{\Omega})\, |\mu'|\, I^C(x,y,z=Z_s,\underline{\Omega}'), \quad \mu > 0. \tag{41g}$$

Thus, the medium is considered non-re-entrant from the top and lateral faces as far as the collided radiation intensity is considered.

The vegetation canopy is divided into cells bounded by $x_{1/2}, x_{3/2}, \ldots, x_{K+1/2}$ (of width Δx), $y_{1/2}, y_{3/2}, \ldots, y_{J+1/2}$ (of width Δy), and $z_{1/2}, z_{3/2}, \ldots, z_{I+1/2}$ (of width Δz). The cross sections σ and σ_s are assumed to be piece-wise constant and can take new values only at the cell-boundaries. Within the cell volume V_{ijk}, bounded by $(x_{k-1/2} < x < x_{k+1/2})$, $(y_{i-1/2} < y < y_{i+1/2})$ and $(z_{i-1/2} < z < z_{i+1/2})$, the cross sections are denoted as σ_{iik} and σ_{siik}.

Introducing the first-order finite-difference approximation for the spatial derivatives in the angle-discretized transport equation for the collided intensity and integrating over the cell volume, yields

$$
\begin{aligned}
-\mu_n \int_j dy \int_k dx\, [I_n(x,y,z_{i+1/2}) - I_n(x,y,z_{i-1/2})] \\
+\eta_n \int_i dz \int_k dx\, [I_n(x,y_{j+1/2},z) - I_n(x,y_{j-1/2},z)] \\
+\xi_n \int_i dz \int_j dy\, [I_n(x_{k+1/2},y,z) - I_n(x_{k-1/2},y,z)] \\
+\sigma_{nijk} \int_i dz \int_j dy \int_k dx\, I_n(x,y,z) \\
= \int_i dz \int_j dy \int_k dx\, J_n(x,y,z),
\end{aligned}
\tag{42}
$$

where J is the total source (S+Q) and, $\int dx$ denotes integration from $x_{k-1/2}$ to $x_{k+1/2}$, and so on. The subscript n denotes the discrete direction of photon travel. Dividing Eq. (42) by the cell volume V_{iik} results in

$$
\begin{aligned}
-\frac{\mu_n}{\Delta z}[I_{njk}(z_{i+1/2}) - I_{njk}(z_{i-1/2})] + \frac{\eta_n}{\Delta y}[I_{nik}(y_{j+1/2}) - I_{nik}(y_{j-1/2})] \\
+\frac{\xi_n}{\Delta x}[I_{nij}(x_{k+1/2}) - I_{nij}(x_{k-1/2})] + \sigma_{nijk} I_{nijk} = J_{nijk}.
\end{aligned}
\tag{43}
$$

In the above, the average radiation intensities over the cell surfaces are

$$I_{njk}(z_{i\pm1/2}) = \frac{1}{\Delta x \Delta y}\int_k dx \int_j dy\, I_n(x,y,z_{i\pm1/2}), \tag{44a}$$

$$I_{nik}(y_{j\pm1/2}) = \frac{1}{\Delta x \Delta z}\int_k dx \int_i dz\, I_n(x,y_{j\pm1/2},z), \tag{44b}$$

$$I_{nij}(x_{k\pm1/2}) = \frac{1}{\Delta z \Delta y}\int_i dz \int_j dy\, I_n(x_{k\pm1/2},y,z). \tag{44c}$$

Similarly, the averages over the cell volume of the specific intensity and the total source are

$$I_{nijk} = \frac{1}{\Delta x \Delta y \Delta z}\int_k dx \int_j dy \int_i dz\, I_n(x,y,z), \tag{45a}$$

$$J_{nijk} = \frac{1}{\Delta x \Delta y \Delta z} \int_k dx \int_j dy \int_i dz \, J_n(x, y, z). \quad (45b)$$

Equation (43) is exact but not closed. To solve for the cell-center angular intensities I_{nijk} and the flows across the three surfaces through which photons can leave the cell volume, three additional relations are required (note that the flows across the three surfaces through which photons enter the cell are known either from the boundary conditions or from previous calculations). The following simple relations can be used for this purpose

$$I_{nijk} \approx 0.5[I_{njk}(z_{i+1/2}) + I_{njk}(z_{i-1/2})], \quad (46a)$$

$$I_{nijk} \approx 0.5[I_{nik}(y_{j+1/2}) + I_{nik}(y_{i-1/2})], \quad (46b)$$

$$I_{nijk} \approx 0.5[I_{nij}(x_{k+1/2}) + I_{nij}(x_{k-1/2})]. \quad (46c)$$

These relations are simple but can lead to negative intensities, in which case remedies must be implemented in the algorithm. The simplest solution is set the offending intensity to zero and proceed with the calculation.

In this manner, the angular and spatial dependence of the transport equation is discretized while insuring that the condition of positivity, symmetry and balance are satisfied. In each octant, the incoming and outgoing flows are identified depending on the sign of the direction cosines in order not to violate the principle of directional evaluation, that is, sweeping in the phase-space along the direction of photon flow only. Using Eqs. (46), the exiting flows can be eliminated to solve for the cell center intensity. A generic equation for the cell center intensity can be written as

$$I_{nijk} = \frac{J_{nijk} + \frac{2\mu_n}{\Delta z} I_{njk}(z_{i\pm1/2}) + \frac{2\eta_n}{\Delta y} I_{nik}(y_{j\pm1/2}) + \frac{2\xi_n}{\Delta x} I_{nij}(x_{k\pm1/2})}{\sigma_{nijk} + \frac{2\mu_n}{\Delta z} + \frac{2\eta_n}{\Delta y} + \frac{2\xi_n}{\Delta x}}. \quad (47)$$

The three flows in the numerator represent the incoming flows across the three faces of the cell and are specific to an octant. The cell center intensity evaluated with Eq. (47) is then used in the relations given in Eqs. (46) to evaluate the three outgoing flows. For example, in octant 1, μ_n, η_n and ξ_n are positive. The three incoming flows are $I_{njk}(z_{i-1/2})$, $I_{nik}(y_{i-1/2})$ and $I_{nij}(x_{k-1/2})$. The outgoing flows to be evaluated are $I_{njk}(z_{i+1/2})$, $I_{nik}(y_{i+1/2})$ and $I_{nij}(x_{k+1/2})$. This phase-space sweeping along the direction of photon travel is embedded in an iteration on the distributed source with appropriate convergence criteria built in. Details on the implementation and acceleration techniques for the iterative procedure can be found in Myneni et al. [1990].

Problem Sets

- **Problem 1.** Derive the one-angle form of the vegetation transport problem.
- **Problem 2**. Derive the one-angle form of the collided and uncollided transport problems.
- **Problem 3**. Derive the analytical solution of the one-angle uncollided transport problem.
- **Problem 4**. Solve the two-stream differential equations for upward F^u and downward F^d fluxes in a vegetation canopy of horizontal leaves

$$\frac{\partial}{\partial L} F^d(L) = \rho_L F^u(L) + (\tau_L - 1) F^d(L),$$

$$\frac{\partial}{\partial L} F^u(L) = \rho_L F^d(L) + (\tau_L - 1) F^u(L),$$

with the boundary conditions

$$F^d(L=0) = F_0^d,$$

$$F^u(L = L_H) = r_S F^d(L = L_H).$$

- **Problem 5**. Show the limiting form of F^u for the case of a very dense canopy.

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Chapter 4 Derivations by Shabanov et al.

Problem 1. Derive the one-angle form of the vegetation transport problem.

Solution. The 1D radiative transfer equation for vegetation canopies is

$$-\mu\frac{\partial}{\partial z}I(z,\underline{\Omega})+u_L(z)G(z,\underline{\Omega})I(z,\underline{\Omega})=\frac{u_L(z)}{\pi}\int_{4\pi}d\underline{\Omega'}\ \Gamma(z,\underline{\Omega'}\to\underline{\Omega})I(z,\underline{\Omega'}). \quad (1)$$

The vertical coordinate z can be changed to cumulative leaf area index L by dividing Eq. (1) with $u_L(z)$, the leaf area density distribution ($L\equiv u_L(z)\cdot z$),

$$-\mu\frac{\partial}{\partial L}I(L,\underline{\Omega})+G(L,\underline{\Omega})I(L,\underline{\Omega})=\frac{1}{\pi}\int_{4\pi}d\underline{\Omega'}\ \Gamma(L,\underline{\Omega'}\to\underline{\Omega})I(L,\underline{\Omega'}). \quad (2)$$

In the following derivations we will assume that the geometry factor and the area scattering phase function are independent of the azimuth angle and cumulative leaf area index, namely,

$$G(L,\underline{\Omega})=G(\mu), \quad (3a)$$

$$\frac{1}{\pi}\Gamma(L,\underline{\Omega'}\to\underline{\Omega})=\frac{1}{\pi}\Gamma(\mu'\to\mu). \quad (3b)$$

In this case the 1D transport equation can be reduced to a one angle problem by averaging Eq. (2) over azimuth angle, φ. The derivations for the first and second items on the left hand side and the remaining item on the right-hand side of Eq. (2) are shown in Eq. (4a)-(4c) below:

$$\frac{1}{2\pi}\int_0^{2\pi}d\varphi\left[-\mu\frac{\partial}{\partial L}I(L,\underline{\Omega})\right]=-\mu\frac{\partial}{\partial L}\frac{1}{2\pi}\int_0^{2\pi}d\varphi\, I(L,\mu,\varphi)=-\mu\frac{\partial}{\partial L}I(L,\mu), \quad (4a)$$

$$\frac{1}{2\pi}\int_0^{2\pi}d\varphi\left[G(L,\underline{\Omega})I(L,\underline{\Omega})\right]=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\, G(L,\varphi,\mu)I(L,\mu,\varphi)=G(\mu)I(L,\mu), \quad (4b)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \left[\int_{4\pi} d\Omega' \, \Gamma(L, \underline{\Omega'} \to \underline{\Omega}) \, I(L, \underline{\Omega'}) \right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu' \int_0^{2\pi} d\varphi' \, \Gamma(L, \mu' \to \mu, \varphi' \to \varphi) \, I(L, \mu', \varphi')$$

$$= 2\pi \int_{-1}^{1} d\mu' \, \Gamma(\mu' \to \mu) \; I(L, \mu'). \tag{4c}$$

In the above we introduced angularly averaged intensity,

$$I(L,\mu) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\varphi \; I(L,\mu,\varphi).$$

Therefore, the one-angle form of 1D RT equation is

$$-\mu \frac{\partial}{\partial L} I(L,\mu) + G(\mu) \, I(L,\mu) = 2\int_{-1}^{1} d\mu' \, \Gamma(\mu' \to \mu) \, I(L,\mu'). \tag{5}$$

The corresponding boundary conditions are

$$I(L = 0, \mu) = I_o \delta(\mu - \mu_o) + I_d(\mu), \quad \mu < 0, \tag{6a}$$

$$I(L = L_H, \mu) = 2\int_{-1}^{0} d\mu' |\mu'| \rho_s(\mu' \to \mu) I(L = L_H, \mu'), \quad \mu > 0, \tag{6b}$$

where

$$I_o(\mu) = \frac{f_{dir}}{|\mu_o|} F_{in}(L = 0) \equiv f_{dir} S_o, \quad S_o \equiv \frac{F_{in}(L = 0)}{|\mu_o|},$$

$$I_d(\mu) = (1 - f_{dir}) d_o(L = 0, \mu) \, F_{in}(L = 0) \equiv (1 - f_{dir}) S_d, \quad S_d \equiv d_o(L = 0, \mu) \, F_{in}(L = 0).$$

Problem 2. Derive the one-angle form of the collided and uncollided transport problems.

Solution. The total intensity, I, can be separated into uncollided, I^0, and collided, I^C, components, namely,

$$I(L,\mu) = I^0(L,\mu) + I^C(L,\mu). \tag{1}$$

Substituting Eq. (1), into the one angle transport problem (cf. Problem 1, Eqs. (5)-(6)), one can split the total problem into the uncollided problem,

$$-\mu\frac{\partial}{\partial L}I^0(L,\mu) + G(\mu)I^0(L,\mu) = 0, \tag{2a}$$

$$I^0(L=0,\mu) = f_{dir}S_o\delta(\mu-\mu_o) + (1-f_{dir})S_d, \quad \mu<0, \tag{2b}$$

$$I^0(L=L_H,\mu) = 2\int_{-1}^{0} d\mu'|\mu'|\rho_s(\mu'\to\mu)I^0(L=L_H,\mu'), \quad \mu>0. \tag{2c}$$

and the collided problem,

$$-\mu\frac{\partial}{\partial L}I^C(L,\mu) + G(\mu)I^C(L,\mu) = 2\int_{-1}^{1} d\mu'\Gamma(L,\mu'\to\mu)\left[I^0(L,\mu') + I^C(L,\mu')\right]. \tag{3a}$$

$$I^C(L=0,\mu) = 0, \quad \mu<0, \tag{3b}$$

$$I^C(L=L_H,\mu) = 2\int_{-1}^{0} d\mu'|\mu'|\rho_s(\mu'\to\mu)I^C(L=L_H,\mu'), \quad \mu>0. \tag{3c}$$

Problem 3. Derive the analytical solution of the one-angle uncollided transport problem.

Solution. The one-angle transport equation for the uncollided radiation is (cf. Problem 2, Eq. (2a))

$$\frac{\partial}{\partial L} I^0(L,\mu) = \frac{G(\mu)}{\mu} I^0(L,\mu) \tag{1}$$

The solution of this equations is

$$I^0(L,\mu) = A(\mu) \cdot \exp\left(\frac{G(\mu)}{\mu} L\right),$$

where coefficient $A(\mu)$ is determined from boundary conditions (cf. Problem 2, Eq. (2b) for $\mu < 0,$ and (2c) for $\mu > 0$).

If $\mu < 0$ (downwelling radiation),

$$I^0(L,\mu) = \left[f_{dir} S_o \delta(\mu - \mu_o) + (1 - f_{dir}) S_d\right] \cdot \exp\left(\frac{G(\mu)}{\mu} L\right). \tag{2a}$$

If $\mu > 0$ (upwelling radiation),

$$\begin{aligned} I^0(L,\mu) &= 2\int_{-1}^{0} d\mu' |\mu'| \rho_s(\mu' \to \mu) [f_{dir} S_o \delta(\mu' - \mu_o) + (1 - f_{dir}) S_d] \exp\left(\frac{G(\mu')}{\mu'}(L - L_H)\right) \\ &= \Bigg[f_{dir} S_o 2|\mu_o| \rho_s(\mu_o \to \mu) \cdot \exp\left(-\frac{G(\mu_o)}{\mu_o} L_H\right) \\ &+ (1 - f_{dir}) S_d 2\int_{-1}^{0} d\mu' |\mu'| \rho_s(\mu' \to \mu) \exp\left(-\frac{G(\mu')}{\mu'} L_H\right)\Bigg] \cdot \exp\left(\frac{G(\mu)}{\mu}(L - L_H)\right). \end{aligned} \tag{2b}$$

Note, in the case of dense canopies

$$I^0(L,\mu)\Big|_{L_H \to \infty} \to 0.$$

Problem 4. Solve the two-stream differential equations for upward F^u and downward F^d fluxes in a vegetation canopy of horizontal leaves

$$\frac{\partial}{\partial L} F^d(L) = \rho_L F^u(L) + (\tau_L - 1) F^d(L),$$

$$-\frac{\partial}{\partial L} F^u(L) = \rho_L F^d(L) + (\tau_L - 1) F^u(L),$$

with the boundary conditions

$$F^d(L = 0) = F_0^d,$$

$$F^u(L = L_H) = r_S F^d(L = L_H).$$

Solution. The two-stream equations in this problem correspond to a homogeneous system of linear differential equations, which can be solved using matrix method. The original system can be rewritten in a matrix form as follows

$$\underline{y}'(L) = \hat{A}\underline{y}(L), \tag{1a}$$

where

$$\underline{y}(L) \equiv \begin{bmatrix} F^d(L) \\ F^u(L) \end{bmatrix}, \quad \underline{y}'(L) \equiv \begin{bmatrix} \frac{\partial}{\partial L} F^d(L) \\ \frac{\partial}{\partial L} F^u(L) \end{bmatrix}, \quad \hat{A} \equiv \begin{bmatrix} (\tau_L - 1) & \rho_L \\ -\rho_L & -(\tau_L - 1) \end{bmatrix}. \tag{1b}$$

If matrix $\hat{A}$ has n=2 independent eigenvectors $\underline{v}_1$ and $\underline{v}_2$ corresponding to eigenvalues λ_1 and λ_2, then the general solution of Eqs. (1a)-(1b) is

$$\underline{y}(L) = C_1 \exp(\lambda_1 L)\underline{v}_1(L) + C_2 \exp(\lambda_2 L)\underline{v}_2(L). \tag{2}$$

The eigenvalues of matrix $\hat{A}$ can be found as follows:

$$\det(\hat{A} - \lambda\hat{I}) = 0$$
$$\Rightarrow -(\tau_L - 1 - \lambda)(\tau_L - 1 + \lambda) + \rho_L^2 = 0$$
$$\Rightarrow \lambda_{1,2} = \pm\lambda \equiv \pm\sqrt{(1-\tau_L)^2 - \rho_L^2}. \qquad (3a)$$

The corresponding eigenvectors are

$$\underline{v}_{1,2}(L) \equiv \begin{bmatrix} -1 \\ (\tau_L - 1 \pm \lambda)/\rho_L \end{bmatrix}. \qquad (3b)$$

Substituting Eq. (3a)-3(b) into Eq. (2) and taking into account definition in given in Eq. (1b), we have

$$\begin{cases} F^d(L) = -C_1 \exp(\lambda L) - C_2 \exp(-\lambda L), \\ F^u(L) = C_1 A \exp(\lambda L) + C_2 B \exp(-\lambda L), \end{cases} \qquad (4a)$$

where

$$A = \frac{\tau_L - 1 - \lambda}{\rho_L}, \quad B = \frac{\tau_L - 1 + \lambda}{\rho_L}. \qquad (4b)$$

Combining Eq. (4a)-(4b) with original boundary conditions, one solves for C_1 and C_2. Therefore, the solution of the two-stream model is

$$F^d(L) = F_0^d \frac{(A + r_S)\exp(-\lambda L) - (B + r_S)\exp(-\lambda[2L_H - L])}{(A + r_S) - (B + r_S)\exp(-2\lambda L_H)},$$

$$F^u(L) = F_0^d \frac{-(A + r_S)B\exp(-\lambda L) + (B + r_S)A\exp(-\lambda[2L_H - L])}{(A + r_S) - (B + r_S)\exp(-2\lambda L_H)},$$

where coefficients, λ, A, and B are given by Eq. (3a) and (4b).

Problem 5. Show the limiting form of F^u for the case of a very dense canopy.

Solution. Recall (cf. Problem 4),

$$F^u(L) = F_0^d \frac{-(A+r_S)B\exp(-\lambda L) + (B+r_S)A\exp(-\lambda[2L_H - L])}{(A+r_s) - (B+r_s)\exp(-2\lambda L_H)},$$

where

$$A = \frac{\tau_L - 1 - \lambda}{\rho_L}, \quad B = \frac{\tau_L - 1 + \lambda}{\rho_L}, \quad \lambda = \sqrt{(1-\tau_L)^2 - \rho_L^2}.$$

Therefore, in the case of dense canopies

$$F^u(L)\Big|_{L_H \to \infty} \to F_0^d \frac{1 - \tau - \sqrt{(1-\tau_L)^2 + \rho_L^2}}{\rho_L} \exp(-L\sqrt{(1-\tau_L)^2 + \rho_L^2}).$$

Chapter 04: Derivations

Derivation 1: Solve the following differential equations for the upward (F^u) and downward (F^d) fluxes for a canopy with horizontal leaves,

$$\frac{\partial}{\partial L} F^d(L) = \rho_L \, F^u(L) + (\tau_L - 1) \, F^d(L) \, , \tag{1a}$$

$$-\frac{\partial}{\partial L} F^u(L) = \rho_L \, F^d(L) + (\tau_L - 1) \, F^u(L) \, . \tag{1b}$$

Derivation 2: Show the explit form for F^u in the limiting case of a very dense canopy.

Derivation 3: Derive expressions for the coefficients K_1^d, K_2^d, K_3^d, K_1^u, K_2^u and K_3^u in the case of a canopy with uniform leaf normal orientation ($G = 0.5$) and bi-Lambertian leaf scattering only [Γ given by Eq. (2.22)]. For definitions of these coefficients see § **IV.8.**

Derivation 4: For the uniform leaf normal and rotationally invariant scattering kernel problem in Derivation 3, for which you derived the coefficients, solve the differential equations for the upward (F^u) and downward (F^d) fluxes. Again, derive the expression for the upward (F^u) flux in the limiting case of a very dense canopy.

Derivation 5: Derive the two-stream transport equations given in Sellers (Canopy reflectance, photosynthesis and transpiration. Int. J. Remote Sens., 6:1335-1372, 1985) starting with the radiative transfer equation.

Chapter 04: Derivation 1 Answer:

For a horizontally homogeneous leaf canopy consisting of horizontal leaves, we have the simplifed differential radiative transfer equations for the upward (F^u) and downward (F^d) fluxes

$$\begin{cases} \frac{\partial}{\partial L} F^d(L) = \rho_L F^u(L) + (\tau_L - 1) F^d(L) \\ -\frac{\partial}{\partial L} F^u(L) = \rho_L F^d(L) + (\tau_L - 1) F^u(L) \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} F^d(L = 0) = F_0^d \\ F^u(L = L_H) = r_s F^d(L = L_H) \end{cases} \tag{2}$$

Our goal is to get the analytical solution of Eq. (1) using four parameters of ρ_L, τ_L, r_s, and F_0^d.

We solve first equation of Eqs.(1) for $F_d(L)$ and substitute into the second equation to obtain the differential equation

$$\frac{d^2 F^d(L)}{dL^2} - \alpha^2 F^d(L) = 0 \tag{3}$$

The differential equaiton

$$\frac{d^2 F^u(L)}{dL^2} - \alpha^2 F^u(L) = 0 \tag{4}$$

can be adjoined by analogy. The constant α in Eqs. (3) and (4) is specified by the relation

$$\alpha = [(1 - \tau_L)^2 - \rho_L^2]^{1/2} \tag{5}$$

General solution of Eqns. (3) and (4) are

$$F^d(L) = C_1 e^{\alpha L} + C_2 e^{-\alpha L} \tag{6}$$

and

$$F^u(L) = C_3 e^{\alpha L} + C_4 e^{-\alpha L} \tag{7}$$

respectively, where C_1, C_2, C_3 and C_4 are constants, which are to be specified by using boundary conditions

$$\begin{cases} F^d(L=0) = F_0^d \\ F^u(L=0) = F_0^u \\ \frac{\partial F^d(L)}{\partial L}(L=0) = \rho_L F_0^u + (\tau_L - 1)F_0^d \\ \frac{\partial F^u(L)}{\partial L}(L=0) = -\rho_L F_0^d - (\tau_L - 1)F_0^u \\ F^u(L=L_H) = r_s F^d(L=L_H) \end{cases} \tag{8}$$

Plugging Eqs. (6) and (7) into (8) we obtain

$$\begin{cases} C_1 + C_2 = F_0^d \\ C_3 + C_4 = F_0^u \\ \alpha(C_1 - C_2) = \rho_L F_0^u + (\tau_L - 1)F_0^d \\ \alpha(C_3 - C_4) = -\rho_L F_0^d - (\tau_L - 1)F_0^u \\ C_3 e^{\alpha L_H} + C_4 e^{-\alpha L_H} = r_s(C_1 e^{\alpha L_H} + C_2 e^{-\alpha L_H}) \end{cases} \tag{9}$$

The solutions of these equations are

$$\begin{cases} C_1 = \frac{\rho_L F_0^u - (1-\tau_L-\alpha)F_0^d}{2\alpha} \\ C_2 = \frac{(1-\tau_L+\alpha)F_0^d - \rho_L F_0^u}{2\alpha} \\ C_3 = \frac{(1-\tau_L+\alpha)F_0^u - \rho_L F_0^d}{2\alpha} \\ C_4 = \frac{\rho_L F_0^d - (1-\tau_L-\alpha)F_0^u}{2\alpha} \\ F_0^u = \frac{b_1 e^{-2\alpha L_H} - b_2}{a_1 - a_2 e^{-2\alpha L_H}} F_0^d \end{cases} \tag{10}$$

where $a_1 = r_s - \frac{1}{A'}$, $a_2 = r_s - A'$, $b_1 = 1 - \frac{r_s}{A'}$, $b_2 = 1 - r_s A'$, and $A' = \frac{1-\tau_L-\alpha}{\rho_L} = \frac{\rho_L}{1-\tau_L+\alpha}$. Then we get our final solutions of Eqns. (1) as following

$$\begin{cases} F^d(L) = F_0^d \frac{a_1 e^{-\alpha L} - a_2 e^{-\alpha(2L_H - L)}}{a_1 - a_2 e^{-2\alpha L_H}} \\ F^u(L) = F_0^d \frac{b_1 e^{-\alpha(2L_H - L)} - b_2 e^{-\alpha L}}{a_1 - a_2 e^{-2\alpha L_H}} \end{cases} \tag{11}$$

Chapter 04: Derivation 2 Answer:

For upward (F^u) flux of a canopy with horizontal leaves, we have

$$F^u(L) = F_0^d \frac{b_1 e^{-\alpha(2L_H - L)} - b_2 e^{-\alpha L}}{a_1 - a_2 e^{-2\alpha L_H}} \tag{1}$$

In the case of a very dense canopy, $L_H \to \infty$. So we have

$$F^u(L) = F_0^d \frac{0 - b_2 e^{-\alpha L}}{a_1 - 0} \tag{2}$$

Note that $a_1 = r_s - \frac{1}{A'}$ and $b_2 = 1 - r_s A'$. So we have

$$\begin{aligned} F^u(L) &= F_0^d \frac{-b_2 e^{-\alpha L}}{a_1} \\ &= F_0^d \frac{(r_s A' - 1) e^{-\alpha L}}{r_s - \frac{1}{A'}} \\ &= F_0^d A' e^{-\alpha L} \end{aligned} \tag{3}$$

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Derivation 3:

Derive expressions for the coefficients K_1^d, K_2^d, K_3^d, K_1^u,K_2^u, K_3^u in the case of a canopy with uniform leaf normal orientation $(G = 0.5)$ and bi_Lambertian leaf scattering only.

Starting from general radiative transfer equation:

$$-\mu\frac{\partial I(L,\Omega)}{\partial L} + G(\Omega)I(L,\Omega) = \frac{1}{\pi}\int_{4\pi}\Gamma(\Omega' \to \Omega)I(L,\Omega')\,d\Omega'. \tag{1}$$

The *normalized scattering phase function* $(1/4\pi)P$ for diffuse scattering only is

$$P(\Omega' \to \Omega) = \frac{4\Gamma(\Omega' \to \Omega)}{\omega_L G(\Omega')}.$$

Rewriting equation (1) as

$$-\mu\frac{\partial I(L,\mu,\phi)}{\partial L} + G(\mu)I(L,\mu,\phi) = \frac{\omega_L}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} P(\mu,\phi;\mu',\phi')G(\mu')I(L,\mu',\phi')\,d\mu'\,d\phi'. \tag{2}$$

In the case of uniformly distributed leaf normals, $G(\Omega)$ is 0.5, and the area scattering phase function and hence $P(\mu,\phi;\mu',\phi')$ is rotationally invariant:

$$\Gamma(\Omega' \to \Omega) = \frac{\omega_L}{3\pi}(\sin\beta - \beta\cos\beta) + \frac{\tau_L}{2}\cos\beta, \tag{3}$$

where $\beta = \arccos(\Omega' \bullet \Omega) = \arccos\{\sin\theta\sin\theta'\cos(\phi - \phi') + \cos\theta\cos\theta'\}$.

For phase function as a function only of the scatttering angle, Chandrasekhar (1960) has shown that the azimuthal integral satifies

$$p(\mu,\mu') = \frac{1}{2\pi}\int_{0}^{2\pi} P(\mu,\phi;\mu',\phi')\,d\phi,$$

With two point quadrature approximation

$$\int_{-1}^{1} p(\mu,\mu')I(L,\mu')\,d\mu' = p(\mu,\mu_1)I(L,\mu_1) + p(\mu,-\mu_1)I(L,-\mu_1),$$

and the additional definition of azimuthal averaged intensity $I(L,\mu)$

$$I(L,\mu) = \int_{0}^{2\pi} I(L,\mu,\phi)\,d\phi,$$

we can get a simpler expression for the hemispherical integration of the rightside term of the RT equation (2),

$$\begin{aligned}
& \frac{\omega_L}{8\pi}\int_{-1}^{0} d\mu \int_{0}^{2\pi} d\phi \int_{-1}^{1}\int_{0}^{2\pi} P(\mu,\phi;\mu',\phi') I(L,\mu',\phi')\, d\mu'\, d\phi' \\
= & \frac{\omega_L}{8\pi}\int_{-1}^{0} d\mu \int_{-1}^{1} d\mu' \int_{0}^{2\pi} I(L,\mu',\phi')\, d\phi' \int_{0}^{2\pi} P(\mu,\phi;\mu',\phi')\, d\phi \\
= & \frac{\omega_L}{4}\int_{-1}^{0} d\mu \int_{-1}^{1} p(\mu,\mu')\, d\mu' \int_{0}^{2\pi} I(L,\mu',\phi')\, d\phi' \\
= & \frac{\omega_L}{4}\int_{-1}^{0} d\mu \int_{-1}^{1} p(\mu,\mu') I(L,\mu')\, d\mu' \\
= & \frac{\omega_L}{4}\int_{-1}^{0} \{p(\mu,\mu_1) I(L,\mu_1) + p(\mu,-\mu_1) I(L,-\mu_1)\}\, d\mu \\
= & \frac{\omega_L}{4}\{I(L,\mu_1)\int_{-1}^{0} p(\mu,\mu_1)\, d\mu + I(L,-\mu_1)\int_{-1}^{0} p(\mu,-\mu_1)\, d\mu\}.
\end{aligned}$$

To further simplify the above term, define the quantity β_i

$$\begin{aligned}
\beta_i &= \frac{1}{2}\int_{-1}^{0} p(\mu_i,\mu')\, d\mu' \\
&= 1 - \frac{1}{2}\int_{0}^{1} p(\mu_i,\mu')\, d\mu',
\end{aligned}$$

and make the following quadrature-like approximation for downwelling flux $F^d(L)$ and upwelling flux $F^u(L)$

$$\begin{aligned}
F^d(L) &= \int_{-1}^{0} |\mu| I(L,\mu)\, d\mu \\
&\simeq \mu_1 I(L,-\mu_1), \\
F^u(L) &= \int_{0}^{1} |\mu| I(L,\mu)\, d\mu \\
&\simeq \mu_1 I(L,\mu_1),
\end{aligned}$$

indicating the following approximation

$$I(L,\mu_1) \simeq F^u(L)/\mu_1,$$

$$I(L,-\mu_1) \simeq F^d(L)/\mu_1.$$

So we get simplified expression,

$$\frac{\omega_L}{8\pi}\{I(L,\mu_1)\int_{-1}^{0} p(\mu,\mu_1)\, d\mu + I(L,-\mu_1)\int_{-1}^{0} p(\mu,-\mu_1)\, d\mu\} = \frac{\omega_L}{2\mu_1}\{(1-\beta_1)F^d(L) + \beta_1 F^u(L)\}.$$

Now, applying the operator $\int_{2\pi^-} d\Omega$ to equation (2),

$$\frac{\partial}{\partial L}\int_{-1}^{0}\{(-\mu) I(L,\mu)\}\, d\phi + \frac{1}{2}\int_{-1}^{0} I(L,\mu)\, d\mu = \frac{\omega_L}{8\pi}\int_{-1}^{0} d\mu \int_{0}^{2\pi} d\phi \int_{-1}^{1}\int_{0}^{2\pi} P(\mu,\phi;\mu',\phi') I(L,\mu',\phi')\, d\mu'\, d\phi',$$

$$\frac{\partial F^d(L)}{\partial L} + \frac{1}{2\mu_1} F^d(L) = \frac{\omega_L}{2\mu_1}\{(1-\beta_1)F^d(L) + \beta_1 F^u(L)\}. \qquad (4)$$

Similarily, for the upwelling flux, applying the operator $\int_{2\pi+} d\Omega$ to equation (2), we can get

$$-\frac{\partial}{\partial L}\int_0^1 \{(\mu)I(L,\mu)\}\, d\phi + \frac{1}{2}\int_0^1 I(L,\mu)\, d\mu = \frac{\omega_L}{8\pi}\int_0^1 d\mu \int_0^{2\pi} d\phi \int_{-1}^1 \int_0^{2\pi} P(\mu,\phi;\mu',\phi') I(L,\mu',\phi')\, d\mu'\, d\phi',$$

$$-\frac{\partial F^d(L)}{\partial L} + \frac{1}{2\mu_1} F^u(L) = \frac{\omega_L}{4}\{I(L,\mu_1)\int_0^1 p(\mu,\mu_1)\, d\mu + I(L,-\mu_1)\int_0^1 p(\mu,-\mu_1)\, d\mu\},$$

$$-\frac{\partial F^d(L)}{\partial L} + \frac{1}{2\mu_1} F^u(L) = \frac{\omega_L}{2\mu_1}\{\beta_1 F^d(L) + (1-\beta_1)F^u\}. \qquad (5)$$

Rearranging equation (4) and (5) as

$$\frac{\partial F^d(L)}{\partial L} = -\frac{1}{2\mu_1}[1-\omega_L(1-\beta_1)]F^d(L) + \frac{\omega_L\beta_1}{2\mu_1}F^u(L), \qquad (6)$$

$$\frac{\partial F^u(L)}{\partial L} = -\frac{\omega_L\beta_1}{2\mu_1}F^d(L) + \frac{1}{2\mu_1}[1-\omega_L(1-\beta_1)]F^u(L). \qquad (7)$$

With Gaussian choice, $\mu_1 = \frac{1}{\sqrt{3}}$.

The only unknown parameter is β_1, so the key then is to derive β_1 with *normalized scattering phase function* infered from equation (3) for uniformly distributed leaf normal,

$$\begin{aligned}
p(\mu,\mu') &= \frac{1}{2\pi}\int_0^{2\pi} P(\mu,\phi;\mu',\phi')\, d\phi \\
&= \frac{4}{2\pi\omega_L 0.5}\int_0^{2\pi}\{\frac{\omega_L}{3\pi}(\sin\beta - \beta\cos\beta) + \frac{\tau_L}{2}\cos\beta\}\, d\phi \\
&= \frac{4}{\pi\omega_L}\int_0^{2\pi}\{\frac{\tau_L}{2} - \frac{\omega_L}{3\pi}\}\cos\beta\, d\phi + \frac{4}{3\pi^2}\int_0^{2\pi}\sin\beta\, d\phi \\
&= \frac{4}{\pi\omega_L}\{\frac{\tau_L}{2} - \frac{\omega_L}{3\pi}\}\int_0^{2\pi}\{\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'\}\, d\phi + \frac{4}{3\pi^2}\int_0^{2\pi}\sin\beta\, d\phi \\
&= \frac{4}{\pi\omega_L}\{\frac{\tau_L}{2} - \frac{\omega_L}{3\pi}\}\cos\theta\cos\theta' + \frac{4}{3\pi^2}\int_0^{2\pi}\sin\beta\, d\phi,
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= \frac{1}{2}\int_{-1}^0 p(\mu_1,\mu')\, d\mu' \\
&= \frac{1}{2}\frac{4}{\pi\omega_L}\{\frac{\tau_L}{2} - \frac{\omega_L}{3\pi}\}\int_{-1}^0 \cos\theta_1\cos\theta'\, d\mu' + \frac{1}{2}\int_{-1}^0 \frac{4}{3\pi^2}\, d\mu' \int_0^{2\pi}\sin\beta\, d\phi \\
&= \frac{2\mu_1}{\pi\omega_L}\{\frac{\tau_L}{2} - \frac{\omega_L}{3\pi}\}\frac{1}{2} + \frac{2}{3\pi^2}\int_{-1}^0 d\mu' \int_0^{2\pi}\sin\beta\, d\phi.
\end{aligned}$$

Since the last term is relatively small, the following approximation could be made

$$\beta_1 \simeq \frac{\mu_1}{\pi\omega_L}\{\frac{\tau_L}{2} - \frac{\omega_L}{3\pi}\}.$$

Derivation 4:

For the uniform leaf normal and rotationally invariant scattering kernel problem in Derivation 3, for which you derived the coefficients, solve the differential equations for the upward (F^u) and downward (F^d) fluxes. Again, derive the expression for the upward (F^u) flux in the limiting case of a very dense canopy.

Solving the linear differential-integral equations (6) and (7) with constant coefficients:

$$\frac{\partial F^d(L)}{\partial L} = C_1 F^d(L) + C_2 F^u(L), \tag{8}$$

$$\frac{\partial F^u(L)}{\partial L} = C_3 F^d(L) + C_4 F^u(L), \tag{9}$$

where

$$\begin{array}{l} C_1 = -\frac{1}{2\mu_1}[1-\omega_L(1-\beta_1)] = -C_4, \\ C_2 = \frac{\omega_L\beta_1}{2\mu_1}, \\ C_3 = -\frac{\omega_L\beta_1}{2\mu_1} = -C_2, \\ C_4 = \frac{1}{2\mu_1}[1-\omega_L(1-\beta_1)], \end{array} \tag{10}$$

and $\beta_1 = \frac{\mu_1}{\pi\omega_L}\{\frac{\tau_L}{2} - \frac{\omega_L}{3\pi}\}$, $\mu_1 = \frac{1}{\sqrt{3}}$.

The boundary conditions are

$$F^d(L=0) = F^0; F^u(L=L_H) = \alpha F^d(L=L_H). \tag{11}$$

where α is surface *albedo* and F^0 is the incident solar flux on the upper boundary of canopy.
From characterization matrix equation

$$\left| \begin{array}{ll} C_1-\lambda & C_2 \\ C_3 & C_4-\lambda \end{array} \right| = 0,$$

we get

$$\begin{aligned} \lambda_1, 2 &= \pm\sqrt{C_2C_3 - C1C_4} \\ &= \pm\frac{1}{2\mu_1}\sqrt{[1-\omega_L(1-\beta_1)]^2 - (\omega\beta_1)^2} \\ &= \pm\frac{1}{2\mu_1}\sqrt{1-\omega}\sqrt{1-\omega+2\omega\beta_1}. \end{aligned}$$

So, $F_d(L)$ and $F^u(L)$ have following gerneral forms of solution,

$$\begin{array}{l} F^d(L) = A_1e^{-\lambda L} + B_1e^{\lambda L}, \\ F^u(L) = A_2e^{-\lambda L} + B_2e^{\lambda L}, \end{array} \tag{12}$$

where $\lambda = \frac{1}{2\mu_1}\sqrt{1-\omega}\sqrt{1-\omega+2\omega\beta_1}$.

Plugging equations (12) to equations (8) and (9), the A_1, B_1, A_2 and B_2 are constrained by

$$\begin{array}{l} (\lambda + C_1)A_1 = -C_2A_2, \\ (\lambda - C_1)A_1 = C_2B_2. \end{array} \tag{13}$$

Combined with two boundary conditions :

$$F^d(L=0) = A_1 + B_1 = F^0, \\ F^u(L=L_H) = A_2 e^{-\lambda L_H} + B_2 e^{\lambda L_H} = \alpha F_d(L=L_H) = \alpha\{A_1 e^{-\lambda L_H} + B_1 e^{\lambda L_H}\}. \quad (14)$$

Solving A_1, B_1, A_2 and B_2 from the above four equations (13) and (14), we get

$$\begin{aligned} A_1 &= \frac{\lambda - C_1 - C_2\alpha}{(\lambda - C_1 - C_2\alpha) + (\lambda + C_1 + C_2\alpha)exp(-2\lambda L_H)} F^0, \\ A_2 &= \frac{(\lambda + C_1)\alpha + C_2}{(\lambda - C_1 - C_2\alpha) + (\lambda + C_1 + C_2\alpha)exp(-2\lambda L_H)} F^0, \\ B_1 &= \frac{\lambda + C_1 + C_2\alpha}{(\lambda + C_1 + C_2\alpha) + (\lambda - C_1 - C_2\alpha)exp(2\lambda L_H)} F^0, \\ B_2 &= \frac{(\lambda - C_1)\alpha - C_2}{(\lambda + C_1 + C_2\alpha) + (\lambda - C_1 - C_2\alpha)exp(2\lambda L_H)} F^0. \end{aligned} \quad (15)$$

λ, C_1 and C_2 all have $\frac{1}{2\mu_1}$, and A_1, B_1, A_2, B_2 are all linear combinations of these three parameters, so $\frac{1}{2\mu_1}$ can be dropped simultaneously when calculating downwelling $F^d(L)$ and upwelling flux $F^u(L)$.

The final solutions of $F^d(L)$ and $F^u(L)$ are expression (12) with the above resolved coefficients.In the case of black soil ($\alpha = 0$),the four coefficients in the solutions can be simplified as

$$\begin{aligned} A_1 &= \frac{\lambda - C_1}{(\lambda - C_1) + (\lambda + C_1)exp(-2\lambda L_H)} F^0, \\ A_2 &= \frac{C_2}{(\lambda - C_1) + (\lambda + C_1)exp(-2\lambda L_H)} F^0, \\ B_1 &= \frac{\lambda + C_1}{(\lambda + C_1) + (\lambda - C_1)exp(2\lambda L_H)} F^0, \\ B_2 &= \frac{-C_2}{(\lambda + C_1) + (\lambda - C_1)exp(2\lambda L_H)} F^0. \end{aligned} \quad (16)$$

In the limiting case of a very dense canopy ($L_H \to \infty$),

$$\begin{aligned} A_1 &= F^0, \\ A_2 &= \frac{(\lambda + C_1)\alpha + C_2}{(\lambda - C_1 - C_2\alpha)} F^0, \\ B_1 &= 0, \\ B_2 &= 0. \end{aligned} \quad (17)$$

From equation (11), the upwelling flux is

$$\begin{aligned} F^u(L) &= A_2 e^{-\lambda L} + B_2 e^{\lambda L}, \\ &= A_2 e^{-\lambda L}, \\ &= \frac{(\lambda + C_1)\alpha + C_2}{(\lambda - C_1 - C_2\alpha)} F^0 e^{-\lambda L}, \\ &= \frac{(\sqrt{1-\omega_L}\sqrt{1-\omega_L+2\omega_L\beta_1})\alpha + \omega_L\beta_1}{\sqrt{1-\omega_L}\sqrt{1-\omega_L+2\omega_L\beta_1} + [1-\omega_L(1-\beta_1)] - \omega_L\beta_1\alpha} F^0 e^{-\sqrt{1-\omega_L}\sqrt{1-\omega_L+2\omega_L\beta_1}L/(2\mu_1)}. \end{aligned}$$

I. One-angle form of radiative transfer equation

The radiative transfer equation for vegetation canopies is

$$-\mu\frac{\partial}{\partial z}I(z,\Omega)+u_L(z)G(z,\Omega)I(z,\Omega)=\frac{u_L(z)}{\pi}\int_{4\pi}\Gamma(z,\Omega'\rightarrow\Omega)I(z,\Omega')d\Omega' \quad . \tag{1}$$

Dividing the above equation through by the *leaf area density distribution* $u_L(z)$, we can change the vertical coordinate from *depth* z to *cumulative leaf area index* L, namely,

$$-\mu\frac{\partial}{\partial L}I(L,\Omega)+G(L,\Omega)I(L,\Omega)=\frac{1}{\pi}\int_{4\pi}\Gamma(L,\Omega'\rightarrow\Omega)I(L,\Omega')d\Omega' \quad . \tag{2}$$

Assume the angular distribution of leaves, $G(L,\Omega)$ and $\Gamma(L,\Omega'\rightarrow\Omega)$, are independent of the *azimuth angle* (ϕ,ϕ'). The above equation can be reduced to a one-angle problem by simply averaging over the *azimuth angle* ϕ:

$$\frac{1}{2\pi}\int_0^{2\pi}\left[-\mu\frac{\partial}{\partial L}I(L,\Omega)\right]d\phi=-\mu\frac{\partial}{\partial L}\frac{1}{2\pi}\int_0^{2\pi}I(L,\phi,\mu)d\phi=-\mu\frac{\partial}{\partial L}I(L,\mu)\ ,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left[G(L,\Omega)I(L,\Omega)\right]d\phi=\frac{1}{2\pi}\int_0^{2\pi}G(L,\phi,\mu)I(L,\phi,\mu)d\phi$$
$$=G(L,\mu)\frac{1}{2\pi}\int_0^{2\pi}I(L,\phi,\mu)d\phi=G(L,\mu)I(L,\mu),$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\int_{4\pi}\Gamma(L,\Omega'\rightarrow\Omega)I(L,\Omega')d\Omega'\right]d\phi=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_{-1}^{1}d\mu'\int_0^{2\pi}\Gamma(L,\phi'\rightarrow\phi,\mu'\rightarrow\mu)I(L,\phi',\mu')d\phi'$$
$$=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_{-1}^{1}d\mu'\Gamma(L,\mu'\rightarrow\mu)\int_0^{2\pi}I(L,\phi',\mu')d\phi'=\int_0^{2\pi}d\phi\int_{-1}^{1}d\mu'\Gamma(L,\mu'\rightarrow\mu)I(L,\mu')$$
$$=2\pi\int_{-1}^{1}d\mu'\Gamma(L,\mu'\rightarrow\mu)I(L,\mu'),$$

where $I(L,\mu)=\frac{1}{2\pi}\int_0^{2\pi}I(L,\phi,\mu)d\phi$.

So one-angle form of equation (2) is given by

$$-\mu\frac{\partial}{\partial L}I(L,\mu)+G(L,\mu)I(L,\mu)=2\int_{-1}^{1}\Gamma(L,\mu'\rightarrow\mu)I(L,\mu')d\mu' . \tag{3}$$

Let assume the angular distribution of leaves, $G(L,\Omega)$ and $\Gamma(L,\Omega'\rightarrow\Omega)$, are independent of the *cumulative leaf area index* L. Equation (3) can be simply reduced to

$$-\mu\frac{\partial}{\partial L}I(L,\mu)+G(\mu)I(L,\mu)=2\int_{-1}^{1}\Gamma(\mu'\rightarrow\mu)I(L,\mu')d\mu', \qquad (4)$$

with the boundary conditions,

$$I(L=0,\mu)=I_0\delta(\mu-\mu_0)+I_d(\mu),\mu<0, \qquad (5)$$

$$I(L=L_H,\mu)=2\int_0^1 d\mu'\,|\mu'|\,\rho_s(\mu'\rightarrow\mu)I(L=L_H,\mu'),\mu>0,\mu'<0, \qquad (6)$$

where

$I_0=\frac{f_{dir}}{|\mu_0|}F_L(L=0)=f_{dir}S_0$, $S_0=F_L(L=0)/|\mu_0|$, and,

$I_d=(1-f_{dir})d_0(L=0,\mu)F_L(L=0)=(1-f_{dir})S_d$, $S_d=d_0(L=0,\mu)F_L(L=0)$.

II. Two-stream approximation of radiative transfer model

1. The $I(L,\mu)$ can be separated into the uncollided and collided components

$$I(L,\mu)=I^o(L,\mu)+I^c(L,\mu). \qquad (7)$$

The transport equation (4) can be split into equation for the uncollided problem

$$-\mu\frac{\partial}{\partial L}I^o(L,\mu)+G(\mu)I^o(L,\mu)=0, \qquad (8)$$

$$I^o(L=0,\mu)=f_{dir}S_0\delta(\mu-\mu_0)+(1-f_{dir})S_d,\mu<0, \qquad (9)$$

$$I^o(L=L_H,\mu)=2\int_0^1 d\mu'\,|\mu'|\,\rho_s(\mu'\rightarrow\mu)I^o(L=L_H,\mu'),\mu>0,\mu'<0, \qquad (10)$$

and collided problem

$$-\mu\frac{\partial}{\partial L}I^c(L,\mu)+G(\mu)I^c(L,\mu)=2\int_{-1}^{1}\Gamma(\mu'\rightarrow\mu)[I^o(L,\mu')+I^c(L,\mu')]d\mu', \qquad (11)$$

$$I^c(L=0,\mu)=0,\mu<0, \qquad (12)$$

$$I^c(L=L_H,\mu)=2\int_0^1 d\mu'\,|\mu'|\,\rho_s(\mu'\rightarrow\mu)I^c(L=L_H,\mu'),\mu>0,\mu'<0. \qquad (13)$$

2. The uncollided problem

Let $K=G(\mu)/\mu$, equation (8) can be written as

$$\frac{\partial}{\partial L} I^o(L,\mu) = K I^o(L,\mu) \ , \tag{14}$$

with the boundary conditions (9) and (10). We get the solution

$$I^o(L,\mu) = [f_{dir} S_0 \delta(\mu - \mu_0) + (1 - f_{dir}) S_d] \exp(KL), \mu < 0 \ , \tag{15}$$

$$\begin{aligned}
I^o(L,\mu) &= 2\int_0^1 d\mu' \,|\,\mu'\,|\, \rho_s(\mu' \to \mu) I^o(L = L_H, \mu') \exp(K(L - L_H)) \\
&= 2\int_0^1 d\mu' \,|\mu'\,|\, \rho_s(\mu' \to \mu)[f_{dir} S_0 \delta(\mu - \mu_0) + (1 - f_{dir}) S_d] \exp(G(\mu') L_H / \mu') \exp(K(L - L_H)) \\
&= 2\,|\,\mu_0\,|\, \rho_s(\mu_0 \to \mu) f_{dir} S_0 \exp(G(\mu_0) L_H / \mu_0) \exp(K(L - L_H)) + 2(1 - f_{dir}) S_d \\
&\quad [\int_0^1 |\,\mu'\,|\, \rho_s(\mu' \to \mu) \Gamma(\mu' \to \mu) \exp(G(\mu') L_H / \mu') d\mu'] \exp(K(L - L_H)), \mu > 0, \mu' < 0 .
\end{aligned} \tag{16}$$

For the dense canopy, we simplify equation (16) as

$$I^o(L,\mu) = 0, \ \mu > 0 . \tag{17}$$

3. The collided problem

The right hand in the collided problem (11) can be rewritten as

$$2\int_{-1}^1 \Gamma(\mu' \to \mu)[I^o(L,\mu') + I^c(L,\mu')] d\mu' = 2\int_{-1}^1 \Gamma(\mu' \to \mu) I^o(L,\mu') d\mu' + 2\int_{-1}^1 \Gamma(\mu' \to \mu) I^c(L,\mu') d\mu' \ .$$

Let consider the dense vegetation for the first component in the right hand

$$\begin{aligned}
\int_{-1}^1 \Gamma(\mu' \to \mu) I^o(L,\mu') d\mu' &= \int_{-1}^1 \Gamma(\mu' \to \mu)[f_{dir} S_0 \delta(\mu' - \mu_0) + (1 - f_{dir}) S_d] \exp(G(\mu') L / \mu') d\mu' \\
&= f_{dir} S_0 \Gamma(\mu_0 \to \mu) \exp(G(\mu_0) L / \mu_0) + (1 - f_{dir}) S_d \int_{-1}^1 \Gamma(\mu' \to \mu) \exp(G(\mu') L / \mu') d\mu' .
\end{aligned}$$

The collided equation (11) becomes

$$\begin{aligned}
-\mu \frac{\partial}{\partial L} I^c(L,\mu) + G(\mu) I^c(L,\mu) &= 2 f_{dir} S_0 \Gamma(\mu_0 \to \mu) \exp(G(\mu_0) L / \mu_0) \\
&+ 2(1 - f_{dir}) S_d \int_{-1}^1 \Gamma(\mu' \to \mu) \exp(G(\mu') L / \mu') d\mu' + 2\int_{-1}^1 \Gamma(\mu' \to \mu) I^c(L,\mu') d\mu' ,
\end{aligned} \tag{18}$$

and with the boundary condition (12) and (13).

4. Approximate solution for diffuse contribution

Let assume the canopy is vertically homogeneous. Multiply equation (18) by $|\mu|/G(\mu)$ and integrate μ over [-1,0],[0,1], respectively,

$$\int_0^1 \left[G(\mu) \frac{|\mu|}{G(\mu)} I^c(L,\mu) \right] d\mu = \int_0^1 \mu I^c(L,\mu) d\mu = I^c(L)\uparrow ,$$

$$\int_{-1}^0 \left[G(\mu) \frac{|\mu|}{G(\mu)} I^c(L,\mu) \right] d\mu = \int_{-1}^0 |\mu| I^c(L,\mu) d\mu = I^c(L)\downarrow ,$$

$$\int_0^1 \left[-\mu \frac{|\mu|}{G(\mu)} \frac{\partial}{\partial L} I^c(L,\mu) \right] d\mu = -\bar{\mu} \frac{\partial}{\partial L} I^c(L)\uparrow ,$$

$$\int_{-1}^0 \left[-\mu \frac{|\mu|}{G(\mu)} \frac{\partial}{\partial L} I^c(L,\mu) \right] d\mu = \bar{\mu} \frac{\partial}{\partial L} I^c(L)\downarrow ,$$

where $\bar{\mu} = \dfrac{\int_0^1 \left[\mu \frac{|\mu|}{G(\mu)} I^c(L,\mu) \right] d\mu}{\int_0^1 \mu I^c(L,\mu) d\mu}$,

$$\int_0^1 \left[\frac{|\mu|}{G(\mu)} 2\int_{-1}^1 \Gamma(\mu' \to \mu) I^o(L,\mu') d\mu' \right] d\mu$$

$$= \int_0^1 \left[\frac{|\mu|}{G(\mu)} 2\int_{-1}^1 \Gamma(\mu' \to \mu) [f_{dir} S_0 \delta(\mu' - \mu_0) + (1 - f_{dir}) S_d] \exp(K' L) d\mu' \right] d\mu$$

$$= 2 f_{dir} S_0 \exp(G(\mu_0) L / \mu_0) \int_0^1 \Gamma(\mu_0 \to \mu) \frac{\mu}{G(\mu)} d\mu$$

$$+ 2(1 - f_{dir}) S_d \int_0^1 \exp(G(\mu') L / \mu') d\mu' \int_0^1 \Gamma(\mu' \to \mu) \frac{\mu}{G(\mu)} d\mu$$

$$= f_{dir} S_0 \exp(G(\mu_0) L / \mu_0) \gamma_0^+(\mu_0) + (1 - f_{dir}) S_d \gamma_d^+(L),$$

Where $\gamma_0^+(\mu_0) = 2\int_0^1 \Gamma(\mu_0 \to \mu) \frac{\mu}{G(\mu)} d\mu,$

$$\gamma_d^+(L) = 2\int_0^1 \exp(G(\mu') L / \mu') d\mu' \int_0^1 \Gamma(\mu' \to \mu) \frac{\mu}{G(\mu)} d\mu$$

$$\int_{-1}^{0}\left[\frac{|\mu|}{G(\mu)}2\int_{-1}^{1}\Gamma\left(\mu' \to \mu\right)I^{o}(L,\mu')d\mu'\right]d\mu$$

$$= -\int_{0}^{-1}\left[\frac{-\mu}{G(\mu)}2\int_{-1}^{1}\Gamma\left(\mu' \to \mu\right)[f_{dir}S_0\delta(\mu' - \mu_0) + (1 - f_{dir})S_d]\exp(K'L)d\mu'\right]d\mu$$

$$= 2f_{dir}S_0\exp(G(\mu_0)L/\mu_0)\int_0^1\Gamma(\mu_0 \to -\mu)\frac{\mu}{G(\mu)}d\mu$$

$$+ 2(1 - f_{dir})S_d\int_0^1\exp(G(\mu')L/\mu')d\mu'\int_0^1\Gamma(\mu' \to -\mu)\frac{\mu}{G(\mu)}d\mu$$

$$= f_{dir}S_0\exp(G(\mu_0)L/\mu_0)\gamma_0^-(\mu_0) + (1 - f_{dir})S_d\gamma_d^-(L)$$

where $\gamma_0^-(\mu_0) = 2\int_0^1\Gamma(\mu_0 \to -\mu)\frac{\mu}{G(\mu)}d\mu$, and,

$$\gamma_d^-(L) = 2\int_0^1\exp(G(\mu')L/\mu')d\mu'\int_0^1\Gamma(\mu' \to -\mu)\frac{\mu}{G(\mu)}d\mu\ .$$

$$\int_{0}^{1}\left[\frac{|\mu|}{G(\mu)}2\int_{-1}^{1}\Gamma\left(\mu' \to \mu\right)I^{c}(L,\mu')d\mu'\right]d\mu$$

$$= 2\int_0^1\frac{\mu}{G(\mu)}d\mu[\int_0^1\Gamma(\mu' \to \mu)I^c(L,\mu')d\mu' + \int_{-1}^0\Gamma(\mu' \to \mu)I^c(L,\mu')d\mu']$$

$$= 2\int_0^1\frac{\mu}{G(\mu)}d\mu[\int_0^1\Gamma(\mu' \to \mu)I^c(L,\mu')d\mu' + 2\int_0^1\Gamma(-\mu' \to \mu)I^c(L,-\mu')d\mu']$$

$$= h_+^+I^c(L)\uparrow + h_+^-I^c(L)\downarrow,$$

where

$$h_+^+ = \frac{2\int_0^1\frac{\mu}{G(\mu)}d\mu\int_0^1\Gamma(\mu' \to \mu)I^c(L,\mu')d\mu'}{\int_0^1\mu I^c(L,\mu)d\mu}\text{, and,}$$

$$h_+^- = \frac{2\int_0^1\frac{\mu}{G(\mu)}d\mu\int_0^1\Gamma(-\mu' \to \mu)I^c(L,-\mu')d\mu'}{\int_0^1\mu I^c(L,\mu)d\mu}\ .$$

$$\int_{-1}^{0}\left[\frac{|\mu|}{G(\mu)}2\int_{-1}^{1}\Gamma\left(\mu' \to \mu\right)I^c(L,\mu')d\mu'\right]d\mu$$

$$= 2\int_{-1}^{0}\frac{-\mu}{G(\mu)}d\mu[\int_{0}^{1}\Gamma(\mu' \to \mu)I^c(L,\mu')d\mu' + 2\int_{-1}^{0}\Gamma(\mu' \to \mu)I^c(L,\mu')d\mu']$$

$$= 2\int_{0}^{1}\frac{\mu}{G(\mu)}d\mu[\int_{0}^{1}\Gamma(\mu' \to -\mu)I^c(L,\mu')d\mu' + 2\int_{0}^{1}\Gamma(-\mu' \to -\mu)I^c(L,-\mu')d\mu']$$

$$= h_{-}^{+}I^c(L)\uparrow + h_{-}^{-}I^c(L)\downarrow,$$

where

$$h_{-}^{+} = \frac{2\int_{0}^{1}\frac{\mu}{G(\mu)}d\mu\int_{0}^{1}\Gamma(\mu' \to -\mu)I^c(L,\mu')d\mu'}{\int_{-1}^{0}|\mu|I^c(L,\mu)d\mu}, \text{ and,}$$

$$h_{-}^{-} = \frac{2\int_{0}^{1}\frac{\mu}{G(\mu)}d\mu\int_{0}^{1}\Gamma(-\mu' \to -\mu)I^c(L,-\mu')d\mu'}{\int_{-1}^{0}|\mu|I^c(L,\mu)d\mu}.$$

So equation (19) can be rewritten as

$$-\bar{\mu}\frac{\partial}{\partial L}I^c(L)\uparrow + I^c(L)\uparrow = h_{+}^{+}I^c(L)\uparrow + h_{+}^{-}I^c(L)\downarrow$$
$$+ f_{dir}S_0\exp(G(\mu_0)L/\mu_0)\gamma_0^{+}(\mu_0) + (1-f_{dir})S_d\gamma_d^{+}(L),$$

$$\bar{\mu}\frac{\partial}{\partial L}I^c(L)\downarrow + I^c(L)\downarrow = h_{-}^{+}I^c(L)\uparrow + h_{-}^{-}I^c(L)\downarrow$$
$$+ f_{dir}S_0\exp(G(\mu_0)L/\mu_0)\gamma_0^{-}(\mu_0) + (1-f_{dir})S_d\gamma_d^{-}(L).$$

5. The two-stream approximation of radiative transfer model

Let treat the individual leaves as isotropic scattering elements, and make the following identification.

$$h_{+}^{+} = h_{-}^{-} = (1-\beta)\omega,$$
$$h_{-}^{+} = h_{+}^{-} = \beta\omega,$$
$$S_0\gamma_0^{+}(\mu_0) = \omega\bar{\mu}\beta_0 G(\mu_0)/\mu_0 = \omega\bar{\mu}K_0\beta_0, K_0 = G(\mu_0)/\mu_0,$$
$$S_0\gamma_0^{-}(\mu_0) = \omega\bar{\mu}(1-\beta_0)G(\mu_0)/\mu_0 = \omega\bar{\mu}K_0(1-\beta_0).$$

The ω is single leaf albedo, β and β_0 are the upscattering parameters for the diffuse and direct beams, respectively. $\bar{\mu}$ is the average inverse diffuse optical depth per unit leaf area and is approximated by $\bar{\mu} = \int_0^1 \mu / G(\mu) d\mu$. The detailed definition of other parameters can be found in related reference.

Here let ignore the diffuse source $S_{d,}$ that is, f_{dir}=1, the above equation becomes

$$-\bar{\mu}\frac{\partial}{\partial L} I^c(L)\uparrow + I^c(L)\uparrow - (1-\beta)\omega I^c(L)\uparrow - \beta\omega I^c(L)\downarrow = f_{dir} S_0 \exp(G(\mu_0)L/\mu_0)\gamma_0^+(\mu_0)$$

$$\Rightarrow -\bar{\mu}\frac{\partial}{\partial L} I^c(L)\uparrow + [1-(1-\beta)\omega]I^c(L)\uparrow - \beta\omega I^c(L)\downarrow = \omega\bar{\mu}K_0\beta_0 \exp(K_0 L), \quad (19)$$

$$\bar{\mu}\frac{\partial}{\partial L} I^c(L)\downarrow + I^c(L)\downarrow - h_-^+ I^c(L)\uparrow - h_-^- I^c(L)\downarrow = f_{dir} S_0 \exp(G(\mu_0)L/\mu_0)\gamma_0^-(\mu_0)$$

$$\Rightarrow \bar{\mu}\frac{\partial}{\partial L} I^c(L)\downarrow + [1-(1-\beta)\omega]I^c(L)\downarrow = \omega\bar{\mu}K_0(1-\beta_0)\exp(K_0 L)\ . \quad (20)$$

Equation (19) and (20) are the two-stream approximation radiative transfer model.

References

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The ES_n Quadrature Scheme by Shabanov et al.

The accuracy of numerical solution of the transport equation with method of discrete ordinates highly depends on selection of quadrature, or method of discretization (cf. Chapter 6). The S_n quadratures were developed by Dr. Carlson to optimize discretization of transport equation in the angular domain [Bass et al., 1986]. The major advantage of the S_n quadratures is a more homogeneous, compared to other quadratures, distribution of nodes over the surface of sphere, which allows in some cases to achieve the required accuracy of numerical calculations with less number of nodes per octant [Bass et al., 1986]. Below we detail one simple and efficient version, called the ES_n quadratures.

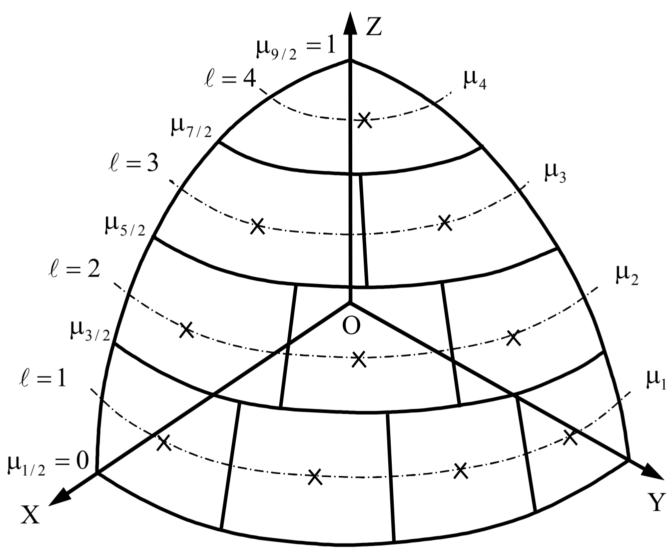

Figure 1. The nodes of ES_n-quadratures for n=8.

The construction the ES_n quadrature, n=2,4,6…, is illustrated for the first octant $\varphi \in [0, \pi/2]$, $\mu \in [0,1]$. The octant is divided into $n(n+2)/8$ parts of equal area, $w_0 = 4\pi/[n(n+2)]$ using latitudes, defined as $\mu = \mu_{\ell+1/2}$, $\ell = 0, 1, ..., n/2$ and longitudes, defined as $\varphi = \varphi_{\ell,\, m+1/2}$, $m = 0, 1, ..., n/2 - \ell + 1$ (Fig. 1). The $\ell -$th layer over μ consists of $n/2 - \ell + 1$ parts of equal area w_0. The total area of the $\ell -$th layer is $\pi w_\ell / 2$, where w_ℓ is the layer width:

$$w_\ell = \frac{4\left[n - 2\ell - 2)\right]}{n(n+2)} = \mu_{\ell+1/2} - \mu_{\ell-1/2}. \tag{1}$$

The coordinates of the boundary $\mu_{\ell\pm1/2}$ and center, $\overline{\mu}_\ell$ of the layer are

$$\mu_{\ell\pm1/2} = 1 - \frac{[n-2\ell+2][n-2(\ell-\pm1)]}{n(n+2)},$$

$$\overline{\mu}_\ell = 1 - \frac{\left[n-2\ell+2\right]^2}{n(n+2)}. \tag{2}$$

The nodes of the quadratures are

$$\mu_\ell = \overline{\mu}_\ell + f\cdot\mu_{\ell-\frac{1}{2}}, \quad \ell = 0,1,\ldots,\frac{n}{2}, \tag{3}$$

$$\varphi_{\ell,m} = \frac{\pi}{2}\left[\frac{2m-1}{n-2\ell+2}\cdot A_n + \frac{1}{2}\cdot(1-A_n)\right], \quad m = 1, 2, \ldots, \frac{n}{2} - \ell + 1. \tag{4}$$

Here m is the sector number in the $\ell-$th layer. Parameters f and A_n are tuning parameters, the value of which are selected to achieve exact evaluation of the following integrals:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \mu^k d\mu, \quad k = 0, 1, 2.$$

Namely,

$$\sum_{\ell=1}^{n/2} w_\ell \mu_\ell^2 = \frac{1}{3}, \quad \sum_{\ell=1}^{n/2}\sum_{m=1}^{n/2-\ell} w_0\,\xi_{\ell,m} = \frac{\pi}{2}\sum_{\ell=1}^{n/2} w_\ell \mu_\ell.$$

In the above, μ_ℓ, ξ_ℓ and η_ℓ are Cartesian coordinates of unit vector $\underline{\Omega}(\theta,\varphi)$

$$\mu_\ell = \cos\theta_\ell, \quad \xi_{\ell,m} = \sqrt{1-\mu_\ell}\cos\varphi_{\ell,m}, \quad \eta_{\ell,m} = \sqrt{1-\mu_\ell}\sin\varphi_{\ell,m}.$$

For the ES_n quadratures the following equalities are valid:

$$w_0 \sum_{\ell,m} \xi_{\ell,m}^2 = w_0 \sum_{\ell,m} \eta_{\ell,m}^2 = w_0 \sum_{\ell,m} \mu_{\ell,m}^2 = \frac{\pi}{6}.$$

The quadratures for the remaining 7 octants are derived using symmetry conditions: $\mu \to -\mu$, $\varphi \to -\varphi$, $\pi/2 - \varphi \to \pi/2 + \varphi$. Note, however, the ES_n quadratures do not posses full symmetry with respect to rotation about coordinate axis X-Y-Z by 90^0, as opposed to the general case of S_n quadratures. Finally, in the case of plane-parallel and spherical geometries the 1-D ES_n quadratures can be used, where weights and nodes over interval [0,1] are specified by Eqs (2) and (3) only.

References

Bass, L., A.M. Voloshenko, A.M., and T.A. Germogenova (1986). *Methods of Discrete Ordinates in Radiation Transport Problems*, Institute of Applied Mathematics, Moscow, (in Russian).