Supplemental Notes for "Finite Order Implications of Common Priors"

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January 2003

These notes provide more details on the Mertens–Zamir [1985] universal belief space and a proof of Corollary 1 of "Finite Order Implications of Common Priors," Lipman [2002]. The discussion on the universal beliefs space is intended to enable a reader to follow the proof of the corollary without requiring further references. It is not intended to be a complete introduction to the subject.

Recall from the text that Θ is the parameter space and is assumed to be compact. For any compact space Z, let $\Delta(Z)$ be the set of probability measures on Z endowed with the weak topology. It is not hard to show that $\Delta(Z)$ is compact.

The universal beliefs space, denoted Ω, is a certain subspace of an infinite product space defined as follows. Let $X_0 = \Theta$ and recursively define $T_{n+1} = \Delta(X_n)$ and $X_{n+1} = X_n \times [T_{n+1}]^I$ where I is the number of players. Let $X = \Theta \times \prod_{n=1}^{\infty} [T_n]^I$. Compactness of Θ implies that X is compact in the product topology.

Mertens–Zamir demonstrate the existence of a subspace of X, denoted Ω, satisfying the following properties. First, there is a set of *types*, T, such that Ω is homeomorphic to $\Theta \times T^I$. Second, T is homeomorphic to $\Delta(\Theta \times T^{I-1})$. Finally, Ω is the largest space with this property. I refer to a point in Ω as a *world*. Intuitively, we can think of a world as a specification of the true value of the unknown parameter for that world and a type for each player. We can think of the type of a player as a probability distribution on Θ and the types of the other players or, equivalently, as a probability distribution on the set of worlds with the property that player i puts probability 1 on his own true type. More precisely, from the homeomorphism between Ω and $\Theta \times T^I$, we can identify i's type at ω. Then from the homeomorphism between T and $\Delta(\Theta \times T^{I-1})$, we identify i's beliefs over $\Theta \times T^{I-1}$. Since i knows his own type, we can write this as a belief over $\Theta \times T^I$ or, via the homeomorphism, over Ω. In short, for any ω and i, we can identify i's beliefs over Ω.

As mentioned in passing above, I use the product topology for X. Because Ω is a subspace of X, it seems natural to topologize Ω by relativizing the topology on X. I follow Mertens–Zamir in using this topology for Ω.

Let $P_i(\omega) \subseteq \Omega$ denote the support of i's beliefs on worlds at ω. A set of worlds $W \subseteq \Omega$ is *belief–closed* if for every i and every $\omega \in W$, $P_i(\omega) \subseteq W$. That is, every world any player believes possible at some $\omega \in W$ is itself contained in W. It is not hard to show that for any ω, there is a smallest belief–closed set containing it, which I denote $\mathcal{B}(\omega)$.[1] I refer to $\mathcal{B}(\omega)$ as the *belief–closed subspace generated by* ω.

As discussed in the text, any partitions model together with any state s in that model uniquely identifies a particular world denoted $\omega(s)$ in the universal beliefs space by the unravelling procedure described earlier. Conversely, any finite belief–closed subspace W of Ω generates a partitions model. More specifically, if W is a finite, belief–closed subspace, we can find a partitions model with the property that the state set in the partitions model is one–to–one with W and each $\omega \in W$ is $\omega(s)$ for some s in the partitions model.[2] When a partitions model $\mathcal{M}$ has this relationship to a belief–closed set W, I say that $\mathcal{M}$ and W are *equivalent*. Similarly, I say that a state s in $\mathcal{M}$ is *equivalent* to a world $\omega \in W$ if $\omega(s) = \omega$.

I say that a world $\omega \in \Omega$ is finite if $\mathcal{B}(\omega)$ is finite. Let Ω_f denote the set of finite worlds. I say that $\omega \in \Omega_f$ is weakly consistent if it is equivalent to a state in a partitions model which is weakly consistent. Let $\Omega_{f,wc}$ denote the set of $\omega \in \Omega_f$ such that ω is weakly consistent. Finally, I will say that a world $\omega \in \Omega_f$ is consistent with common priors if it is equivalent to a state in a partitions model which satisfies the common prior assumption. Let $\Omega_{f,cp}$ denote the set of $\omega \in \Omega_f$ such that ω is consistent with common priors. Note that ω can only be consistent with common priors if it is weakly consistent. Hence $\Omega_{f,cp} \subseteq \Omega_{f,wc}$. For any set Z, let $\text{cl}(Z)$ denote its closure.

[1] Obviously, Ω itself is belief–closed, so every world is contained in at least one belief–closed set. It is easy to see that the intersection of an arbitrary collection of belief–closed sets is belief–closed. Hence the intersection of the family of belief–closed sets containing ω is the smallest belief–closed set containing ω. For clarity, I emphasize that $\mathcal{B}(\omega)$ need not be a *minimal* belief–closed set. That is, it may contain a proper subset which is belief–closed. If so, the proper subset must not contain ω.

[2] One can extend this converse to infinite W if one replaces partitions with σ–fields and allows for infinite S. However, this issue is irrelevant for my purposes. See Brandenburger and Dekel [1993] or Tan and Werlang [1988] for details.

Theorem 1 in the text yields

Lemma 1 $\mathrm{cl}(\Omega_{f,cp}) = \mathrm{cl}(\Omega_{f,wc})$.

Proof. Obviously, since $\Omega_{f,cp} \subseteq \Omega_{f,wc}$, we have $\mathrm{cl}(\Omega_{f,cp}) \subseteq \mathrm{cl}(\Omega_{f,wc})$. For the converse, fix any $\omega \in \Omega_{f,wc}$. Let s be a state in partitions model $\mathcal{M}$ which is equivalent to ω. Such an s and $\mathcal{M}$ must exist because ω is finite. By definition, s is weakly consistent. By Theorem 1 in the text, for any N, we can find a partitions model satisfying common priors and a state s^N in that model such that the n^{th} order beliefs at s^N are the same as those at s for all $n \leq N$. Let $\omega^N = \omega(s^N)$. Clearly, $\omega^N \in \Omega_{f,cp}$. Because ω^N is the world generated by s^N, ω^N has the same parameter value as ω and has the same n^{th} order beliefs for each player as ω for all $n \leq N$. Hence $\omega^N \to \omega$ as $N \to \infty$. Hence $\Omega_{f,wc} \subseteq \mathrm{cl}(\Omega_{f,cp})$, so $\mathrm{cl}(\Omega_{f,wc}) \subseteq \mathrm{cl}(\Omega_{f,cp})$. ▮

Also,

Lemma 2 $\mathrm{cl}(\Omega_{f,wc}) = \mathrm{cl}(\Omega_f)$.

Proof. Analogously to the above, it is sufficient to show that $\Omega_f \subseteq \mathrm{cl}(\Omega_{f,wc})$. So fix any $\omega \in \Omega_f$. Since ω is finite, it is equivalent to a state in a partitions model. Let s^* and $\mathcal{M}$ be such a state and partitions model. For each finite N, construct a new partitions model $\mathcal{M}^N$ as follows. S, f, and the partitions in $\mathcal{M}^N$ are the same as those in $\mathcal{M}$. The prior for i, μ_i^N, is defined by

$$\mu_i^N(s \mid \pi_i(s')) = \frac{1}{N}\ \frac{1}{\#\pi_i(s')} + \frac{N-1}{N}\mu_i(s \mid \pi_i(s'))$$

for $s \in \pi_i(s')$ where $\#$ denotes cardinality. Let ω^N be the world consistent with s^* in model $\mathcal{M}^N$. Obviously, ω^N is finite. It is easy to see that for every player j and event E in $\mathcal{M}^N$, $E \cap B_j^0(E) = \emptyset$ so for $\mathcal{M}^N$ the event τ is equal to S. Hence s^* in $\mathcal{M}^N$ is weakly consistent so ω^N is weakly consistent. Hence $\omega^N \in \Omega_{f,wc}$. It is easy to see that for any event $E \subseteq S$, $\mu_i^N(E \mid \pi_i(s^*)) \to \mu_i(E \mid \pi_i(s^*))$ as $N \to \infty$. Hence $\omega^N \to \omega$. ▮

Finally, Mertens and Zamir's Theorem 3.1 implies

Lemma 3 $\mathrm{cl}(\Omega_f) = \Omega$.

Hence we obtain the corollary stated in the text:

Corollary 1 *The closure of the set of finite worlds consistent with common priors is* Ω.

References

[1] Brandenburger, A., and E. Dekel, "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, **59**, February 1993, pp. 189–198.

[2] Lipman, B., "Finite Order Implications of Common Priors," Boston University working paper, December 2002.

[3] Mertens, J.-F., and S. Zamir, "Formalization of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, **14**, 1985, pp. 1–29.

[4] Tan, T., and S. Werlang, "On Aumann's Notion of Common Knowledge — An Alternative Approach," working paper, 1988.