Optimal Allocation with Costly Verification

Online Appendix

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1 Reduction

In this Appendix, we show that we can reduce the principal's problem to the choice of $(\mathbf{p}, \mathbf{q})$ functions as in the text. We begin with an arbitrary mechanism which could have multiple stages of cheap talk statements by the agents and checking by the principal, where who can speak and which agents are checked depend on past statements and the results from past checks, finally culminating in the allocation of the good, perhaps to no one. Think of such a dynamic mechanism as a game in extensive form between the agents and the principal where the principal is committed in advance to his strategy. The principal's actions specify decisions about which agent or agents to check at various points and, ultimately, which (if any) to allocate the good to. Fix such a dynamic mechanism, deterministic or otherwise, and any equilibrium, say σ, in pure or mixed strategies. We show that the principal's payoff in this mechanism can be duplicated or improved by the appropriate choice of $(\mathbf{p}, \mathbf{q})$ functions.

There are three key steps to the reduction. The first step is a version of the Revelation Principle appropriate to this setting which shows that, without loss of generality, we can restrict attention to truth–telling equilibria in direct mechanisms. In the second step,

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we use our assumption of perfect verification to show that we can restrict attention to mechanisms where all checks the principal carries out are done simultaneously. Finally, the last step identifies two simple properties of the optimal allocation.

To show the first step, we take the equilibrium, σ, of the original mechanism and construct a new mechanism as follows. Each player i reports a type $t_i \in \mathcal{T}_i$. Given a vector of reports $\mathbf{t}$, the principal simulates what would happen in the original mechanism when the agents play the strategies $\sigma(\mathbf{t})$. That is, they play as they would have in the equilibrium if the true types were $\mathbf{t}$. As the principal simulates the mechanism, he may check some agents' types. If he gets all the way through the simulation without any checks revealing that some agent's report is false, he allocates the good as in the simulation.

Suppose that some checks reveal that one or more agents have lied. If more than one agent must have lied, then the principal allocates the good arbitrarily. Since we will only be interested in truth–telling equilibria, only unilateral lies will be relevant for incentives.

So suppose that checks reveal only that agent i has lied — that is, the outcome of checks are consistent with the reports of all agents $j \neq i$. If agent i has another move after this point, the principal can no longer simulate the mechanism using the strategy $\sigma_i(t_i)$. By definition, the information set corresponding to this later move is one that could not have been reached if i really were type t_i, so no strategy for type t_i would say anything about this information set. To continue the simulation, the principal chooses any feasible strategy for i from this point forward. Again, he completes the simulation and then allocates the good according to the result of the simulation.

It is easy to see that truth telling is an equilibrium of this game. Fix any player i of type t_i and assume that all agents $j \neq i$ report truthfully. Then i's payoff from reporting truthfully as well is exactly the same as in the equilibrium of the original mechanism. His payoff from reporting any other type is exactly the same as his payoff to a certain deviation in the original mechanism. Hence the fact that the original strategies formed an equilibrium implies that truth telling is a best reply. Clearly, the principal's payoff in the truth telling equilibrium is the same as in the original mechanism.

Next, we show that given our assumption that verification is perfect — that is, if the principal checks agent i, he learns i's true type — it is without loss of generality to assume that the principal carries out whatever checks he does all at once. In other words, there is no need for him to decide whether to check an agent based on the outcome of earlier checks.

To see this, again, fix any mechanism and any equilibrium. Now we construct a direct mechanism as follows. If the reported type profile is $\mathbf{t}$, then the principal computes the probability distribution over which agents would be checked in the equilibrium of the original mechanism given that the true types are $\mathbf{t}$. He then randomizes over the set of agents to check using this probability distribution, but carries out these checks simultaneously rather than sequentially. For example, if in the original mechanism, he would have checked agent 1, then randomized 50–50 over whether to check agent 2, the principal randomizes 50–50 over checking just agent 1 or checking both 1 and 2 simultaneously. Similarly, if the principal would have checked agent 1 and then only checked 2 if he learned 1 had some type other than t_1, then he just checks agent 1 since this is what would happen conditional on the types being $\mathbf{t}$.

If what the principal observes from the checks is consistent with what he would have seen in the equilibrium (that is, for every agent j he checks, he sees that j's type is t_j), then he allocates the good exactly as he would have done in the equilibrium after these observations. If there is only a single player, say i, who is found to have type $t'_i \neq t_i$, then the allocation of the good is the same as it would have been in the original equilibrium if the type profile were $(t'_i, \mathbf{t}_{-i})$, players $j \neq i$ used their equilibrium strategies, and player i deviated to the equilibrium strategy of type t_i. Finally, the allocation is arbitrary if the principal learns that two or more players have types different from their reports.

As before, truth telling is an equilibrium of this game. For any player i, consider the best reply of type t_i to truth–telling by the other agents. Just as before, i's payoff from reporting truthfully is the same as in the equilibrium of the original mechanism. Just as before, his payoff to reporting any other type is the same as his payoff to a certain deviation in the original mechanism. Therefore, the fact that we began with equilibrium strategies for the original mechanism implies that a best reply for t_i is to report t_i. Clearly, the payoff for the principal is the same as before.

With imperfect verification, sequential checking procedures may be needed. However, this is just a matter of computing the statistical tools available to the principal. That is, a sequential procedure for checking gives the principal a certain probability distribution over observations and costs as a function of the true types. One can simply determine the set of such conditional distributions and treat the principal as picking among them. That is, we can translate the problem into one with *different* stochastic verification technologies, corresponding to what can be done in the sequential environment and have the principal use these simultaneously.

For the third step, we give two simple but useful properties of the optimal allocation. First, given that we focus on truth telling equilibria, all situations in which agent i's report is checked and found to be false are off the equilibrium path. The specification of the mechanism for such a situation cannot affect the incentives of any agent $j \neq i$ since agent j will expect i's report to be truthful. Thus the specification only affects agent i's incentives to be truthful. Since we want i to have the strongest possible incentives to report truthfully, we may as well assume that if i's report is checked and found to be false, then the good is given to agent i with probability 0. Hence we can further reduce the complexity of a mechanism to specify which agents are checked and which agent receives the good as a function of the reports, where the latter applies only when the checked reports are accurate.

Finally, any agent's incentive to reveal his type is unaffected by the possibility of being checked in situations where he does not receive the object regardless of the outcome of the check. That is, if an agent's report is checked even when he would not receive the object if found to have told the truth, his incentives to report honestly are not affected. Since checking is costly for the principal, this means that if the principal checks an agent, then (if he is found to have been honest), he must receive the object with probability 1.

Therefore, we can think of the mechanism as specifying two probabilities for each agent: the probability he is awarded the object without being checked and the probability he is awarded the object conditional on a successful check. As in the text, we let $q_i(\mathbf{t})$ denote the probability i is awarded the object conditional on a successful check and let $p_i(\mathbf{t})$ be the total probability i is awarded the object.

2 Proof of Theorem 4

The proof of Theorem 4 proceeds with a series of lemmas. Throughout we write the distribution of t_i as a measure μ_i. Recall that we have assumed this measure is absolutely continuous with respect to Lebesgue measure on the interval $\mathcal{T}_i \subset \mathbf{R}$. We let μ be the product measure on the product Borel field of $\mathcal{T}$. For any $S \subseteq \mathcal{T}$, let

$$S(t_i) = \{\mathbf{t}_{-i} \in \mathcal{T}_{-i} \mid (t_i, \mathbf{t}_{-i}) \in S\}$$

denote the t_i *fiber* of S. Let S_i denote the projection of S on $\mathcal{T}_i$, and S_{-ij} the projection on $\prod_{k \notin \{i,j\}} \mathcal{T}_k$.

We begin with a technical lemma.[1]

Lemma 1. *Given any Borel measurable $S \subset \mathbf{R}^i$ with $\mu(S) > 0$, there exists $S^* \subseteq S$ with $\mu(S^*) = \mu(S)$ such that the following holds. First, for every i and every $t_i \in \mathcal{T}_i$, the measure of every fiber is strictly positive. That is, $\mu_{-i}(S(t_i)) > 0$ for all i and all $t_i \in \mathcal{T}_i$. Second, for all i, the projection on i of S^*, S_i^*, is measurable.*

Moreover, given any j, there exists $\varepsilon > 0$ and $S^{**} \subseteq S$ with $\mu(S^{**}) > 0$ such that the following holds. First, for every $i \neq j$ and every $t_i \in \mathcal{T}_i$, the measure of every fiber is strictly positive. That is, $\mu_{-i}(S^{**}(t_i)) > 0$. Second, for every $t_j \in S_j^{**}$, the fiber $S^{**}(t_j)$

[1] We thank Benjy Weiss for suggesting the idea of the following proof.

has measure bounded below by ε. That is, $\mu_{-j}(S^{**}(t_j)) > \varepsilon$. Finally, for all i, S_i^{**}, the projection on i of S^{**}, is measurable.

Proof. We first prove this for $I = 2$, and then show how to extend it to $I > 2$. So, to simplify notation for the first step, denote by x and y the two dimensions. Fix a Borel measurable S with $\mu(S) > 0$. We need to show that there is an equal measure subset of S, S^*, such that all fibers of S^* have strictly positive measure and all projections of S^* are measurable. So we need to show (1) $\mu_x(S^*(y)) > 0$ for all y, (2) $\mu_y(S^*(x)) > 0$ for all x, and (3) the projections of S^* are measurable.

First, we observe that if all the fibers have strictly positive measure, then the projections are measurable. To see this, note that the function $f : X \to \mathbf{R}$ given by $f(x) = \mu_y(S^*(x))$ is measurable by Fubini's Theorem. Hence the set $\{x \mid \mu_y(S^*(x)) > 0\}$ is measurable. But this is just the projection on the first dimension if the fiber has positive measure. An analogous argument applies to the y coordinate.

Let S^1 denote the set S after we delete all x fibers with μ_y measure zero. That is, $S^1 = S \cap [\{x \mid \mu_y(S(x)) > 0\} \times \mathbf{R}]$. We know that S^1 is measurable, has the same measure as S (by Fubini, because we deleted only fibers of zero measure), all its x fibers have strictly positive y measure, and its projection on x is measurable.

We do not know that the projection of S^1 on y is measurable nor that the y fibers have strictly positive x measure. Let S^2 denote the set S^1 after we delete all y fibers with μ_x measure zero. That is, $S^2 = S^1 \cap [\{y \mid \mu_x(S^1(y)) > 0\} \times \mathbf{R}]$. We know that S^2 is measurable with the same measure as S^1, that its projection on y is measurable, and all its y fibers have strictly positive y measure.

Again, we do not know that its projection on x is measurable nor that the x fibers have strictly positive y measure. But at this step we do know that the set of x fibers that have zero measure is contained in a set of measure zero. Put differently,

$$\mu_x \left\{x \mid \mu_y \left(S^2(x)\right) > 0\right\} = \mu_x \left(S_x^1\right) = \mu_x \left\{x \mid \mu_y \left(S^1 (x)\right) > 0\right\}. \quad (1)$$

To see this, suppose not. Then

$$\mu_x \left\{x \mid \mu_y \left(S^2 (x)\right) > 0\right\} < \mu_x \left\{x \mid \mu_y \left(S^1 (x)\right) > 0\right\}$$

as

$$\left\{x \mid \mu_y \left(S^2 (x)\right) > 0\right\} \subseteq \left\{x \mid \mu_y \left(S^1 (x)\right) > 0\right\}.$$

Let

$$\Delta = \left\{x \mid \mu_y \left(S^{\mathbf{1}} (x)\right) > 0\right\} \setminus \left\{x \mid \mu_y \left(S^2 (x)\right) > 0\right\}.$$

If $\mu(\Delta) > 0$, then

$$\begin{aligned}\mu\left(S^1\right) &= \int_{\{x|\mu_y(S^1(x))>0\}} \mu_y\left(S^1(x)\right)\mu_x(dx) \\ &= \int_{\{x|\mu_y(S^2(x))>0\}} \mu_y\left(S^1(x)\right)\mu_x(dx) + \int_\Delta \mu_y\left(S^1(x)\right)\mu_x(dx) \\ &> \int_{\{x|\mu_y(S^2(x))>0\}} \mu_y\left(S^2(x)\right)\mu_x(dx) \\ &= \mu\left(S^2\right)\end{aligned}$$

as $S^1(x) \supseteq S^2(x)$ and $\mu(\Delta) > 0$. But this contradicts $\mu(S^2) = \mu(S^1)$. Hence equation (1) holds.

Finally, let S^3 denote S^2 after we delete all x fibers with μ_y measure zero. That is, $S^3 = S^2 \cap [\{x \mid \mu_y(S^2(x)) > 0\} \times \mathbf{R}]$. We know that S^3 is measurable with the same measure as S^2, that its projection on x is measurable, and that all its x fibers have strictly positive y measure. But now we also know that all the y fibers have strictly positive x measure, since in going from S^2 to S^3, we deleted a set of x's contained in a set of zero measure. Hence each y fiber has the same measure as before.

We now extend this to $I > 2$. For brevity, we only describe the extension to $I = 3$, the more general result following the same lines. Denote the coordinates by x, y, and z. Consider a set S with $\mu(S) > 0$. We show there exists $S^* \subseteq S$ such that $\mu_{yz}(S^*(x)) > 0$ for all $x \in S^*_x$, and similarly for all $y \in S^*_y$ and all $z \in S^*_z$.

From the case of $I = 2$, we know there exists $S^1 \subseteq S$ with $\mu(S^1) = \mu(S)$ such that for all $x \in S^1_x$, we have $\mu_{yz}(S^1(x)) > 0$ and for all $(y, z) \in S^1_{yz}$, we have $\mu_x(S^1((y, z))) > 0$. Applying $I = 2$ result again to the set S^1_{yz}, we have $G \subseteq S^1_{yz}$ with $\mu_{yz}(G) = \mu_{yz}(S^1_{yz})$ such that for all $y \in G_y$, we have $\mu_z(G(y)) > 0$ and for all $z \in G_z$, we have $\mu_y(G(z)) > 0$. (Note that this implies that $\mu_{yz}(G) > 0$.)

Now define

$$S^2 = S^1 \cap (\mathbf{R} \times G) = \left\{(x, y, z) \mid (x, y, z) \in S^1 \text{ and } (y, z) \in G\right\}.$$

Since $G \subseteq S^1_{yz}$ and $\mu_{yz}(G) = \mu_{yz}(S^1_{yz})$, we have $\mu(S^2) = \mu(S^1)$. Clearly, $S^2_y = G_y$ and $S^2_z = G_z$. Fix any $y \in S^2_y$. Since $y \in G_y$, we have $\mu_z\{z \mid (y, z) \in G\} > 0$. Since $G \subseteq S^1_{yz}$ for every $(y, z) \in G$, we have $\mu_x(S^2(y, z)) = \mu_x(S^1(y, z)) > 0$. By Fubini's Theorem, $\mu_{xz}(S^2(y)) = \int_{z \in G(y)} \mu_x(S^2(y, z))\mu_z(dz)$ and hence $\mu_{xz}(S^2(y)) > 0$. A similar argument implies that for all $z \in S^2(z)$, we have $\mu_{xy}(S^2(z)) > 0$. However, we do not know that for every $x \in S^2_x$, we have $\mu_{y,z}(S^2(x)) > 0$. Hence we now define the set S^3 by

$$S^3 = S^2 \cap \left(\left\{x \mid \mu_{yz}(S^2(x)) > 0\right\} \times \mathbf{R}^2\right).$$

Clearly, S^3_x is measurable and we have $\mu_{yz}(S^3(x)) > 0$ for every $x \in S^3_x$. Furthermore, $S^3 \subseteq S^2 \subseteq S^1$ and hence $S^3_x \subseteq S^1_x$. In fact, $\mu(S^3) = \mu(S^2) = \mu(S^1)$ implies $\mu(S^3_x) = \mu(S^1_x)$. To see this, suppose not. Then $\mu_x(S^3_x) < \mu_x(S^1_x)$. Since for each $x \in S^1_x$, we have $\mu_{yz}(S^1(x)) > 0$, we obtain that $\mu(S^3) < \mu(S^1)$, a contradiction.

We claim that S^3 satisfies the properties stated in the first part of the lemma. That is, (1) S^3_y and S^3_z are measurable, (2) for all $y \in S^3_y$, we have $\mu_{x,z}(S^3(y)) > 0$, and (3) for all $z \in S^3_z$, we have $\mu_{x,y}(S^3(z)) > 0$. Consider an element $y \in S^2_y$. We have seen that for all $z \in G(y)$, we have $\mu_x(S^2(y,z)) > 0$. Since our construction of S^3 removes from S^2 a set of elements x in S^2_x that is contained in a set of measure zero, we must have $\mu_x(S^3(y,z)) = \mu_x(S^2(y,z)) > 0$. Hence $S^3_y = S^2_y$ and for every $y \in S^3_y$, we have $\mu_{xz}(S^3(y)) = \mu_{xz}(S^2(y)) > 0$. A similar argument establishes that $S^3_z = S^2_z$ and that for $z \in S^3_z$, we have $\mu_{xy}(S^3(y)) > 0$. By defining $S^* = S^3$, we obtain a set S^* with the properties claimed in the first part of the lemma.

It remains to prove the "moreover" claim. This follows from a similar argument where in defining S^1, we remove all x's whose fibers do not have probability at least ε for an appropriately chosen ε. We provide the proof for the case $I = 2$. The proof for $I > 2$ is similar.

Note that

$$\{x \mid \mu_y(S(x)) > 0\} = \bigcup_{n=1}^{\infty} \{x \mid \mu_y(S(x)) > 1/n\}.$$

Since $\mu_x(\{x \mid \mu_y(S(x)) > 0\}) > 0$, there exists $\hat{n}$ such that $\mu_x(\{x \mid \mu_y(S(x)) > 1/\hat{n}\}) > 0$. Define $\varepsilon = 1/\hat{n}$ and define $S^1 = S \cap (\{x \mid \mu_y(S(x)) > \varepsilon\} \times \mathbf{R})$.

The rest of the argument is essentially identical to the argument given in the proof of the first part of the lemma. Specifically, we know that S^1_x is measurable and that for every $x \in S^1_x$, we have $\mu_y(S^1(x)) > \varepsilon$. Define

$$S^2 = S^1 \cap \left(\left\{y \mid \mu_x(S^1(y)) > 0\right\} \times \mathbf{R}\right)$$

$$S^3 = S^2 \cap \left(\left\{x \mid \mu_y(S^2(x)) > \varepsilon\right\} \times \mathbf{R}\right).$$

We have $S^3 \subseteq S^2 \subseteq S^1$. Fubini's Theorem implies that $\mu(S^2) = \mu(S^1)$ which in turn implies that

$$\mu_x(\left\{x \mid \mu_y(S^2(x)) > \varepsilon\right\}) = \mu_x(\left\{x \mid \mu_y(S^1(x)) > \varepsilon\right\}).$$

To see this, suppose not. Then $S^2 \subseteq S^1$ and the fact that $\mu_y(S^1(x)) > \varepsilon$ for all $x \in S^1_x$ implies that $\mu(S^2) < \mu(S^1)$, a contradiction.

Since

$$\left\{x \mid \mu_y(S^2(x)) > \varepsilon\right\} = \left\{x \mid \mu_y(S^3(x)) > \varepsilon\right\} = S^3_x,$$

we see that $\mu_x(S^1_x) = \mu_x(S^3_x)$. Hence in moving from S^2 to S^3, the set of x's that is deleted is contained in a set of measure zero. Since for all $y \in S^2_y$, we have $\mu_x(S^2(y)) > 0$, we see that $S^3_y = S^2_y$ and that $\mu_x(S^3(y)) > 0$ for all $y \in S^3_y$. Thus the set S^3 satisfies all the properties stated in the second paragraph of the lemma. ∎

For the remaining lemmas, fix $\mathbf{p}$ and φ that maximize

$$\mathrm{E}_{\mathbf{t}}\left[\sum_i [p_i(\mathbf{t})(t_i - c_i) + \varphi_i c_i]\right] = \sum_i \left\{\mathrm{E}_{t_i}\left[\hat{p}_i(t_i)(t_i - c_i)\right] + \varphi_i c_i\right\}$$

subject to $\sum_i p_i(\mathbf{t}) \leq 1$ for all $\mathbf{t}$ and $\hat{p}_i(t_i) \geq \varphi_i \geq 0$ for all i and t_i where $\hat{p}_i(t_i) = \mathrm{E}_{\mathbf{t}_{-i}} p_i(\mathbf{t})$. As explained in Section V, the optimal $\mathbf{q}$ will then be any feasible $\mathbf{q}$ satisfying $\hat{q}_i(t_i) = \hat{p}_i(t_i) - \varphi_i$ for all i and t_i where $\hat{q}_i(t_i) = \mathrm{E}_{\mathbf{t}_{-i}} q_i(\mathbf{t})$.

Lemma 2. *There is a set $\mathcal{T}' \subseteq \mathcal{T}$ with $\mu(\mathcal{T}') = 1$ such that the following hold:*

1. For each i, if $t_i < c_i$ and $t_i \in \mathcal{T}'_i$, then $\hat{p}_i(t_i) = \varphi_i$.
2. For each $\mathbf{t} \in \mathcal{T}'$, if $t_i > c_i$ for some i, then $\sum_j p_j(\mathbf{t}) = 1$.
3. For any $\mathbf{t} \in \mathcal{T}'$, if $\hat{p}_i(t_i) > \varphi_i$ for some i, then $\sum_j p_j(\mathbf{t}) = 1$.

Proof. Proof of 1. If $t_i < c_i$, then the objective function is strictly decreasing in $\hat{p}_i(t_i)$. Obviously, reducing $\hat{p}_i(t_i)$ makes the other constraints easier to satisfy. Since we improve the objective function and relax the constraints by reducing $\hat{p}_i(t_i)$, we must have $\hat{p}_i(t_i) = \varphi_i$ at the optimum. This completes the proof of part 1. Since we only characterize optimal mechanisms up to sets of measure zero, we abuse notation, and redefine $\mathcal{T}$ to equal a measure 1 subset of $\mathcal{T}$ on which property 1 is satisfied, and whose projections are measurable (which exists by Lemma 1).

Proof of 2. Suppose not. Then there exists an agent i and a set $\hat{\mathcal{T}}$ with positive measure such that for every $\mathbf{t} \in \hat{\mathcal{T}}$, we have $t_i > c_i$ and yet $\sum_j p_j(\mathbf{t}) < 1$. Define an allocation function $\mathbf{p}^*$ by

$$p^*_j(\mathbf{t}) = \begin{cases} p_j(\mathbf{t}), & \text{if } j \neq i \text{ or } \mathbf{t} \notin \hat{\mathcal{T}} \\ 1 - \sum_{j \neq i} p_j(\mathbf{t}), & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathbf{p}^*$ satisfies all the constraints and improves the objective function, a contradiction.

Proof of 3. Suppose to the contrary that we have a positive measure set of $\mathbf{t}$ such that $\sum_j p_j(\mathbf{t}) < 1$ but for each $\mathbf{t}$, there exists some i with $\hat{p}_i(t_i) > \varphi_i$. Then there exists i and a positive measure set of $\mathbf{t}$ such that for each $\mathbf{t}$, we have $\sum_j p_j(\mathbf{t}) < 1$ and $\hat{p}_i(t_i) > \varphi_i$.

From part 1, we know that for all $t_i \in \mathcal{T}_i$ with $\hat{p}_i(t_i) > \varphi_i$ we have $t_i > c_i$. Hence from part 2, the mechanism is not optimal, a contradiction. ∎

Abusing notation, redefine $\mathcal{T}$ to equal a measure 1 subset of $\mathcal{T} \setminus \mathcal{T}'$ whose projections are measurable (which exists by Lemma 1) on which all the properties of Lemma 2 are satisfied everywhere.

Lemma 3. *There is a set* $\mathbf{t}' \subseteq \mathcal{T}$ *with* $\mu(\mathcal{T}') = 1$ *such that for any* $\mathbf{t} \in \mathcal{T}'$, *if* $t_i - c_i > t_j - c_j$ *and* $\hat{p}_j(t_j) > \varphi_j$, *then* $p_j(\mathbf{t}) = 0$.

Proof. Suppose not. Then we have a positive measure set S such that for all $\mathbf{t} \in S$, $t_i - c_i > t_j - c_j$, $\hat{p}_j(t_j) > \varphi_j$, and $p_j(\mathbf{t}) > 0$. Hence there exists $\alpha > 0$ and $\varepsilon > 0$ such that $\mu(\hat{S}) > 0$ where

$$\hat{S} = \{\mathbf{t} \in \mathcal{T} \mid t_i - c_i - (t_j - c_j) \geq \alpha,\ \hat{p}_j(t_j) \geq \varphi_j + \varepsilon,\ \text{and } p_j(\mathbf{t}) \geq \varepsilon\}.$$

Define $\mathbf{p}^*$ by

$$p_j^*(\mathbf{t}) = \begin{cases} p_k(\mathbf{t}), & \text{for } k \neq i, j \text{ or } \mathbf{t} \notin \hat{S} \\ p_j(\mathbf{t}) - \varepsilon, & \text{for } k = j \text{ and } \mathbf{t} \in \hat{S} \\ p_i(\mathbf{t}) + \varepsilon, & \text{for } k = i \text{ and } \mathbf{t} \in \hat{S}. \end{cases}$$

Since $p_j(\mathbf{t}) \geq \varepsilon$ for all $\mathbf{t} \in \hat{S}$, we have $p_k^*(\mathbf{t}) \geq 0$ for all k and $\mathbf{t}$. Obviously, $\sum_k p_k^*(\mathbf{t}) = \sum_k p_k(\mathbf{t})$, so the constraint that the p_k's sum to less than one must be satisfied.

Turning to the lower bound constraint on the $\hat{p}_k$'s, obviously, for $k \neq j$, we have $\hat{p}_k^*(t_k) \geq \hat{p}_k(t_k) \geq \varphi_k$, so the constraint is satisfied for all $k \neq j$ and all t_k. For any t_j, either $\hat{p}_j^*(t_j) = \hat{p}_j(t_j)$ or

$$\hat{p}_j^*(t_j) = \hat{p}_j(t_j) - \varepsilon\mu_{-j}(\hat{S}\,(t_j)) \geq \hat{p}_j(t_j) - \varepsilon.$$

But for each t_j for which $\hat{p}_j^*(t_j) \neq \hat{p}_j(t_j)$, we have $\hat{p}_j(t_j) \geq \varphi_j + \varepsilon$, so

$$\hat{p}_j^*(t_j) \geq \varphi_j + \varepsilon - \varepsilon = \varphi_j.$$

Hence for every k and every t_k, we have $\hat{p}_k^*(t_k) \geq \varphi_k$. Therefore, $\mathbf{p}^*$ is feasible given φ.

Finally, the change in the principal's payoff in moving from $\mathbf{p}$ to $\mathbf{p}^*$ is

$$\mu(\hat{S})\varepsilon \left[\mathrm{E}(t_i - c_i \mid \mathbf{t} \in \hat{S}) - \mathrm{E}(t_j - c_j \mid \mathbf{t} \in \hat{S})\right] \geq \mu(\hat{S})\varepsilon\alpha > 0.$$

Hence $\mathbf{p}$ could not have been optimal, a contradiction. ∎

Thus the set $S^0 = \{\mathbf{t} \in \mathcal{T} \mid t_i - c_i - (t_j - c_j) > 0,\ \hat{p}_j(t_j) > \varphi_j,\ \text{and } p_j(\mathbf{t}) > 0\}$ has measure zero. Abusing notation, redefine $\mathcal{T}$ to equal a measure 1 subset of $\mathcal{T} \setminus S^0$ whose projections are measurable (which exists by Lemma 1).

Lemma 4. *There is a set of measure one $\mathcal{T}' \subseteq \mathcal{T}$ such that for all $\mathbf{t}', \mathbf{t}'' \in \mathcal{T}'$ such that $t'_j = t''_j$, $p_j(\mathbf{t}') > 0$, $\hat{p}_i(t'_i) > \varphi_i$, $t''_i < t'_i$, and $\hat{p}_i(t''_i) > \varphi_i$, we have $p_i(\mathbf{t}'') = 0$.*

The idea that underlies the proof is simple. Consider two profiles $\mathbf{t}'$ and $\mathbf{t}''$ that have the properties stated in the lemma. That is, $t'_j = t''_j$, $p_j(\mathbf{t}') > 0$, $\hat{p}_i(t'_i) > \varphi_i$, $t''_i < t'_i$, and $\hat{p}_i(t''_i) > \varphi_i$. Suppose the claim is false, so that $p_i(\mathbf{t}'') > 0$. Clearly, there is some $\varepsilon > 0$ such that $p_j(\mathbf{t}') > \varepsilon$, $\hat{p}_i(t'_i) > \varphi_i + \varepsilon$, and $\hat{p}_i(t''_i) > \varphi_i + \varepsilon$, and $p_i(\mathbf{t}'') > \varepsilon$. For simplicity, assume $\mu(\mathbf{t}') = \mu(\mathbf{t}'') = \delta > 0$. (The formal proof will extend the argument to the case that μ is a general atomless probability measure.) Consider the following transfer of allocation probabilities between agents i and j. For the profile $\mathbf{t}'$, increase $p_i(\mathbf{t}')$ by ε and decrease $p_j(\mathbf{t}')$ by ε. For the profile $\mathbf{t}''$, decrease $p_i(\mathbf{t}'')$ by ε and increase $p_j(\mathbf{t}'')$ by ε. Let $\mathbf{p}^*$ denote the resulting probability function. It is easy to see that $\mathbf{p}^*$ satisfies all the constraints. Also, it increases the value of the objective function because the net effect of the transfers is to move a probability $\varepsilon\delta$ of allocating the object from type t''_i to type t'_i where $t'_i > t''_i$. This argument is not sufficient for the general proof, of course, since μ is atomless, implying that we must change $\mathbf{p}$ on a positive measure set of types to have an effect.

Proof. Given any rational number α and any $t'_j \in \mathcal{T}_j$, let

$$\hat{A}_{-j}(\alpha, t'_j) = \{\mathbf{t}'_{-j} \in \mathcal{T}_{-j} \mid t'_i > \alpha,\ \hat{p}_i(t'_i) > \varphi_i,\ \text{and } p_j(\mathbf{t}') > 0\}$$

$$\hat{B}_{-j}(\alpha, t'_j) = \{\mathbf{t}'_{-j} \in \mathcal{T}_{-j} \mid t'_i < \alpha,\ \hat{p}_i(t'_i) > \varphi_i,\ \text{and } p_i(\mathbf{t}') > 0\}.$$

$$\hat{C}_j(\alpha) = \{t'_j \in \mathcal{T}_j \mid \mu_{-j}(\hat{A}_{-j}(\alpha, t'_j)) > 0,\ \mu_{-j}(\hat{B}_{-j}(\alpha, t'_j)) > 0\}.$$

Also let

$$\tilde{A}_{-j}(\alpha, t'_j, \varepsilon, \delta) = \{\mathbf{t}'_{-j} \in \mathcal{T}_{-j} \mid t'_i > \alpha + \delta,\ \hat{p}_i(t'_i) > \varphi_i + \varepsilon,\ \text{and } p_j(\mathbf{t}') > \varepsilon\}$$

$$\tilde{B}_{-j}(\alpha, t'_j, \varepsilon, \delta) = \{\mathbf{t}'_{-j} \in \mathcal{T}_{-j} \mid t'_i < \alpha - \delta,\ \hat{p}_i(t'_i) > \varphi_i + \varepsilon,\ \text{and } p_i(\mathbf{t}') > \varepsilon\}.$$

and

$$\begin{aligned}\tilde{A}(\alpha, \varepsilon, \delta) &= \{\mathbf{t} \in \mathcal{T} \mid t_i > \alpha + \delta,\ \hat{p}_i(t_i) > \varphi_i + \varepsilon,\ \text{and } p_j(\mathbf{t}) > \varepsilon\} \\ &= \bigcup_{t'_j \in \mathcal{T}_j} \{t'_j\} \times \tilde{A}_{-j}(\alpha, t'_j, \varepsilon, \delta)\end{aligned}$$

$$\begin{aligned}\tilde{B}(\alpha, \varepsilon, \delta) &= \{\mathbf{t} \in \mathcal{T} \mid t_i < \alpha - \delta,\ \hat{p}_i(t_i) > \varphi_i + \varepsilon,\ \text{and } p_i(\mathbf{t}) > \varepsilon\} \\ &= \bigcup_{t'_j \in \mathcal{T}_j} \{t'_j\} \times \tilde{B}_{-j}(\alpha, t'_j, \varepsilon, \delta)\end{aligned}$$

$$\tilde{C}_j(\alpha, \varepsilon, \delta) = \{t'_j \in \mathcal{T}_j \mid \mu_{-j}(\tilde{A}_{-\mathbf{j}}(\alpha, t'_j, \varepsilon, \delta)) > 0,\ \mu_{-j}(\tilde{B}_{-j}(\alpha, t'_j, \varepsilon, \delta)) > 0\}.$$

Finally let

$$\bar{A}\left(\alpha,\varepsilon,\delta\right)=\bigcup_{t_j\in\tilde{C}_j(\alpha,\varepsilon,\delta)}\{t_j\}\times\tilde{A}_{-j}(\alpha,t_j,\varepsilon,\delta)=\left(\tilde{C}_j\left(\alpha,\varepsilon,\delta\right)\times\mathcal{T}_{-j}\right)\cap\tilde{A}\left(\alpha,\varepsilon,\delta\right)$$
$$\bar{B}\left(\alpha,\varepsilon,\delta\right)=\bigcup_{t_j\in\tilde{C}_j(\alpha,\varepsilon,\delta)}\{t_j\}\times\tilde{B}_{-j}(\alpha,t_j,\varepsilon,\delta)=\left(\tilde{C}_j\left(\alpha,\varepsilon,\delta\right)\times\mathcal{T}_{-j}\right)\cap\tilde{B}\left(\alpha,\varepsilon,\delta\right)$$

Measurability of all the sets defined above follows from standard arguments.

We now show that for every rational number α, we have $\mu_j(\hat{C}_j(\alpha))=0$. So suppose not. Fix the rational α for which it fails. Then there must be $\varepsilon>0$ and $\delta>0$ such that $\mu_j(\tilde{C}_j(\alpha,\varepsilon,\delta))>0$. For notational simplicity, we drop the arguments $\alpha,\varepsilon,\delta$ in the next step of the argument as they are fixed in this step.

Define $\mathbf{p}^*$ as follows. For $k\neq i,j$ and any $\mathbf{t}$, $p_k^*(\mathbf{t})=p_k(\mathbf{t})$. Also, for any $\mathbf{t}\notin\bar{A}\cup\bar{B}$ and all k, $p_k^*(\mathbf{t})=p_k(\mathbf{t})$. For $\mathbf{t}\in\bar{A}$,

$$p_j^*(\mathbf{t})=p_j(\mathbf{t})-\varepsilon\mu_{-j}\left(\tilde{B}_{-j}\left(t_j\right)\right)\quad\text{and}\quad p_i^*(\mathbf{t})=p_i(\mathbf{t})+\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right).$$

For $\mathbf{t}\in\bar{B}$,

$$p_j^*(\mathbf{t})=p_j(\mathbf{t})+\varepsilon\mu_{-j}\left(\tilde{A}_{-j}\left(t_j\right)\right)\quad\text{and}\quad p_i^*(\mathbf{t})=p_i(\mathbf{t})-\varepsilon\mu_{-j}\left(\tilde{A}_{-j}\left(t_j\right)\right).$$

For $\mathbf{t}\in\bar{A}$, we have $p_j(\mathbf{t})\geq\varepsilon$, while for $\mathbf{t}\in\bar{B}$, $p_i(\mathbf{t})\geq\varepsilon$. Hence $\mathbf{p}^*$ satisfies non–negativity. Clearly, for any $\mathbf{t}$, $\sum_k p_k^*(\mathbf{t})=\sum_k p_k(\mathbf{t})$, so $\mathbf{p}^*$ satisfies the constraint that the sum of the p_i's is less than 1.

Obviously, for $k\neq i,j$, we have $\hat{p}_k^*(t_k)=\hat{p}_k(t_k)\geq\varphi_k$. So the lower bound constraint on $\hat{p}_k(t_k)$ holds for all t_k for all $k\neq i,j$. Clearly, for any t_j such that $p_j^*(\mathbf{t})\geq p_j(\mathbf{t})$ for all t_{-j}, we have $\hat{p}_j(t_j)\geq\varphi_j$. Otherwise, we have

$$\hat{p}_j^*(t_j)=\hat{p}_j(t_j)-\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\mu_{-j}(\tilde{A}(t_j))+\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\mu_{-j}\left(\tilde{A}_{-j}\left(t_j\right)\right).$$

So $\hat{p}_j^*(t_j)=\hat{p}_j(t_j)$. Hence $\hat{p}_j^*(t_j)\geq\varphi_j$ for all $t_j\in\mathcal{T}_j$.

For any t_i such that $p_i^*(\mathbf{t})\geq p_i(\mathbf{t})$ for all $\mathbf{t}_{-i}$, we have $\hat{p}_i(t_i)\geq\varphi_i$. So consider t_i such that $p_i^*(\mathbf{t})<p_i(\mathbf{t})$ for some $\mathbf{t}_{-i}$. Then there must be $\mathbf{t}_{-i}$ such that $\mathbf{t}\in\bar{B}$. Hence $\hat{p}_i(t_i)\geq\varphi_i+\varepsilon$. So

$$\begin{aligned}\hat{p}_i^*(t_i)&=\hat{p}_i(t_i)+\varepsilon\mu_{-i}\left(\{t_{-i}\mid(t_i,t_{-i})\in\bar{A}\}\right)\mu_{-j}\left(\tilde{B}_{-j}\left(t_j\right)\right)\\&\quad-\varepsilon\mu_{-i}\left(\{t_{-i}\mid(t_i,t_{-i})\in\bar{B}\}\right)\mu_{-j}\left(\tilde{A}_{-j}\left(t_j\right)\right)\\&\geq\hat{p}_i(t_i)-\varepsilon\\&\geq\varphi_i+\varepsilon-\varepsilon=\varphi_i.\end{aligned}$$

Hence the lower bound constraint for $\hat{p}_i$ also holds everywhere.

Finally, the change in the principal's payoff from switching to $\mathbf{p}^*$ from $\mathbf{p}$ is

$$\begin{aligned}
&\int_{\mathbf{t}\in\bar{A}} \left\{\left[(t_i - c_i)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right] - \left[(t_j - c_j)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right]\right\} \mu(d\mathbf{t}) \\
&\qquad + \int_{\mathbf{t}\in\bar{B}} \left\{\left[-(t_i - c_i)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right] + \left[(t_j - c_j)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right]\right\} \mu(d\mathbf{t}) \\
&\quad = \int_{\tilde{C}_j} \left(\int_{\tilde{A}_{-j}(t_j)} \left\{\left[(t_i - c_i)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right] - \left[(t_j - c_j)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right]\right\} \mu_{-j}(d\mathbf{t}_{-j})\right. \\
&\qquad \left. + \int_{\tilde{B}_{-j}(t_j)} \left\{\left[-(t_i - c_i)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right] + \left[(t_j - c_j)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right]\right\} \mu_{-j}(d\mathbf{t}_{-j})\right) \mu_j(dt_j).
\end{aligned}$$

Note that $(t_j - c_j)\varepsilon\mu_{-j}(\tilde{A}_{-j}(t_j))$ and $(t_j - c_j)\varepsilon\mu_{-j}(\tilde{B}_{-j}(t_j))$ are functions only of t_j, not $\mathbf{t}_{-j}$. Hence we can rewrite the above as

$$\begin{aligned}
&\int_{\tilde{C}_j} \left(-\left[(t_j - c_j)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right] \int_{\tilde{A}_{-j}(t_j)} \mu_{-j}(d\mathbf{t}_{-j})\right) \mu_j(dt_j) \\
&\quad + \int_{\tilde{C}_j} \left(\left[(t_j - c_j)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right] \int_{\tilde{B}_{-j}(t_j)} \mu_{-j}(d\mathbf{t}_{-j})\right) \mu_j(dt_j) \\
&\quad + \int_{\tilde{C}_j} \left(\int_{\tilde{A}_{-j}(t_j)} \left[(t_i - c_i)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right] \mu_{-j}(d\mathbf{t}_{-j})\right) \mu_j(dt_j) \\
&\quad - \int_{\tilde{C}_j} \left(\int_{\tilde{B}_{-j}(t_j)} \left[(t_i - c_i)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right] \mu_{-j}(d\mathbf{t}_{-j})\right) \mu_j(dt_j)
\end{aligned}$$

The first two lines sum to zero. For the last two lines, recall that $\mathbf{t} \in \bar{A}$ implies $t_i \geq \alpha + \delta$, while $\mathbf{t} \in \bar{B}$ implies $t_i \leq \alpha - \delta$. Hence the last two lines sum to at least

$$\begin{aligned}
&\int_{\tilde{C}_j} \left(\int_{\tilde{A}_{-j}(t_j)} \left[(\alpha + \delta)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right] \mu_{-j}(d\mathbf{t}_{-j})\right) \mu_j(dt_j) \\
&\quad - \int_{\tilde{C}_j} \left(\int_{\tilde{B}_{-j}(t_j)} \left[(\alpha - \delta)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right] \mu_{-j}(d\mathbf{t}_{-j})\right) \mu_j(dt_j) \\
&= \int_{\tilde{C}_j} \left[(\alpha + \delta)\varepsilon\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\right] \mu_j(dt_j) \\
&\quad - \int_{\tilde{C}_j} \left[(\alpha - \delta)\varepsilon\mu_{-j}\left(\tilde{A}_{-j}(t_j)\right)\mu_{-j}\left(\tilde{B}_{-j}(t_j)\right)\right] \mu_j(dt_j) \\
&> 0.
\end{aligned}$$

Hence the payoff difference for the principal between $\mathbf{p}^*$ and $\mathbf{p}$ is strictly positive. Hence $\mathbf{p}$ could not have been optimal, a contradiction.

This establishes that for every rational α, $\mu_j(\hat{C}_j(\alpha)) = 0$.

To complete the proof, let

$$\hat{A}_j(\alpha) = \{t_j \in \mathcal{T}_j \mid \mu_{-j}(\hat{A}_{-j}(\alpha, t_j)) = 0\}$$

and

$$\hat{B}_j(\alpha) = \{t_j \in \mathcal{T}_j \mid \mu_{-j}(\hat{B}_{-j}(\alpha, t_j)) = 0\}.$$

It is easy to see that for any α, $\hat{A}_j(\alpha) \cup \hat{B}_j(\alpha) \cup \hat{C}_j(\alpha) = \mathcal{T}_j$. Let

$$\begin{aligned} A(\alpha) &= \bigcup_{t_j \in \hat{A}_j(\alpha)} \{t_j\} \times \hat{A}_{-j}(\alpha, t_j) \\ &= \{\mathbf{t} \in \mathcal{T} \mid t_i > \alpha,\ \hat{p}_i(t_i) > \varphi_i,\ \text{ and } p_j(\mathbf{t}) > 0\} \cap \left[\hat{A}_j\left(\alpha\right) \times \mathcal{T}_{-j}\right] \end{aligned}$$

$$\begin{aligned} B(\alpha) &= \bigcup_{t_j \in \hat{B}_j(\alpha)} \{t_j\} \times \hat{B}_{-j}(\alpha, t_j) \\ &= \{\mathbf{t} \in \mathcal{T} \mid t_i < \alpha,\ \hat{p}_i(t_i) > \varphi_i,\ \text{ and } p_i(\mathbf{t}) > 0\} \cap \left[\hat{B}_j\left(\alpha\right) \times \mathcal{T}_{-j}\right] \end{aligned}$$

$$C(\alpha) = \bigcup_{t_j \in \hat{C}_j(\alpha)} \{t_j\} \times \mathcal{T}_{-j}$$

and

$$D(\alpha) = A(\alpha)\ \cup\ B(\alpha)\ \cup\ C(\alpha).$$

Once again measurability of the sets just defined is straightforward.

Note that $\mu\left(A\left(\alpha\right)\right) = 0$, since

$$\begin{aligned} \mu(A(\alpha)) &= \int_{\hat{A}_j(\alpha)} \mu_{-j}(A_{-j}(\alpha, t_j))\mu_j(dt_j) \\ &= \int_{\hat{A}_j(\alpha)} \mu_{-j}(\hat{A}_{-j}(\alpha, t_j))\mu_j(dt_j) \\ &= 0, \end{aligned}$$

where the last equality follows from $\mu_{-j}(\hat{A}_{-j}(\alpha, t_j)) = 0$ for all $t_j \in \hat{A}_j(\alpha)$. Similarly, $\mu(B(\alpha)) = 0$. Also, $\mu(C(\alpha)) = \mu_j(\hat{C}_j(\alpha))\mu_{-j}(\mathcal{T}_{-j})$ which is 0 by the first step. Hence $\mu(D(\alpha)) = 0$.

Let $S = \cup_{\alpha \in \mathbb{Q}} D(\alpha)$ where $\mathbb{Q}$ denotes the rationals. Clearly $\mu(S) = 0$.

To complete the proof, suppose that, contrary to our claim, there exists $\mathbf{t}', \mathbf{t}'' \in \mathcal{T} \setminus S$ such that $p_j(\mathbf{t}') > 0$, $\hat{p}_i(t_i') > \varphi_i$, $t_i'' < t_i'$, and $\hat{p}_i(t_i'') > \varphi_i$, but $p_i(t_j', t_i'', t_{-ij}'') > 0$. Obviously, there exists a rational α such that $t_i'' < \alpha < t_i'$. Hence $(t_i', t_{-ij}') \in \hat{A}_{-j}(\alpha, t_j')$ and $(t_i'', t_{-ij}'') \in \hat{B}_{-j}(\alpha, t_j')$. Since $\mathbf{t}'$ is not in S, we know that $\mathbf{t}' \notin A(\alpha)$, implying that

$t'_j \notin \hat{A}_j(\alpha)$. Similarly, since $\mathbf{t}''$ is not in S, we have $\mathbf{t}'' \notin B(\alpha)$, so $t'_j \notin \hat{B}_j(\alpha)$. Similarly, $\mathbf{t}' \notin C(\alpha)$, implying $t'_j \notin C_j(\alpha)$. But $\hat{A}_j(\alpha) \cup \hat{B}_j(\alpha) \cup \hat{C}_j(\alpha) = \mathcal{T}_j$, a contradiction. ▌

Abusing notation, define $\mathcal{T}$ to be a measure one subset of $\mathbf{t}'$ whose projections are measurable and such that for all $\mathbf{t}', \mathbf{t}'' \in \mathcal{T}$ for which $t'_j = t''_j$, $p_j(\mathbf{t}') > 0$, $\hat{p}_i(t'_i) > \varphi_i$, $t''_i < t'_i$, and $\hat{p}_i(t''_i) > \varphi_i$, we have $p_i(\mathbf{t}'') = 0$.

Lemma 5. *There is a set of measure one T' such that if $\hat{p}_j(t_j) = \varphi_j$, $\hat{p}_i(t_i) > \varphi_i$, and*

$$\mu_i\left(\{t'_i \in \mathcal{T}_i \mid t'_i < t_i \text{ and } \hat{p}_i(t'_i) > \varphi_i\}\right) > 0,$$

then $p_j(\mathbf{t}) = 0$.

Proof. Let

$$\mathcal{T}_i^* = \{t_i \in \mathcal{T}_i \mid \hat{p}_i(t_i) > \varphi_i \text{ and } \mu_i(\{t'_i \mid \hat{p}_i(t'_i) > \varphi_i \text{ and } t'_i < t_i\}) > 0\}.$$

To see that $\mathcal{T}_i^*$ is measurable. note that

$$\mathcal{T}_i^* = \tilde{T}_i^* \bigcap \{t_i \in \mathcal{T}_i \mid \hat{p}_i(t_i) > \varphi_i\}$$

where

$$\tilde{\mathcal{T}}_i^* = \{t_i \in \mathcal{T}_i \mid \mu_i(\{t'_i \mid \hat{p}_i(t'_i) > \varphi_i \text{ and } t'_i < t_i\}) > 0\}.$$

Since $\tilde{\mathcal{T}}_i^*$ is an interval (i.e., $\hat{t}_i \in \tilde{\mathcal{T}}_i^*$ and $t''_i > \hat{t}_i$ implies $t''_i \in \tilde{\mathcal{T}}_i^*$), it is measurable. Hence $\mathcal{T}_i^*$ is the intersection of two measurable sets and so is measurable.

Suppose the claim of the lemma is not true. Then there exists $\varepsilon > 0$ such that $\mu(S) > 0$ and such that S has measurable projections where

$$S = \{\mathbf{t} \in \mathcal{T} \mid t_i \in \mathcal{T}_i^*,\ \hat{p}_j(t_j) = \varphi_j,\ \hat{p}_i(t_i) > \varphi_i + \varepsilon, \text{ and } p_j(\mathbf{t}) \geq \varepsilon\},$$

and where we use Lemma 1 and take an equal measure subset if necessary.

Since $\mu(S) > 0$, we must have $\mu_i(S_i) > 0$ and hence $\mu_i(\mathcal{T}_i^*) > 0$ since $S_i \subseteq \mathcal{T}_i^*$. Choose measurable sets $L_i, M_i, U_i \subset S_i$ such that the following hold. First, all three sets have strictly positive measure. Second, $\sup L_i < \inf M_i$ and $\sup M_i < \inf U_i$. (Think of U, M, and L as standing for "upper," "middle," and "lower" respectively.) Third, there is an $\varepsilon' > 0$ such that $\mu(\hat{S}) > 0$ where $\hat{S}$ is defined as follows. Let

$$S'' = \bigcup_{t_i \in U_i} \{t_i\} \times \{t_{-i} \in \mathcal{T}_{-i} \mid (t_i, t_{-i}) \in S\} = \{\mathbf{t} \in \mathcal{T} \mid t_i \in U_i,\ \hat{p}_j(t_j) = \varphi_j, \text{ and } p_j(\mathbf{t}) \geq \varepsilon\}.$$

Clearly $\mu(S'') > 0$. By Lemma 1, there exists a positive measure set $\hat{S} \subset S''$ and a number $\varepsilon' > 0$ satisfying the following. First, $\hat{S}$ has strictly positive measure fibers. That is, for all i and all t_i, $\mu_{-i}(\hat{S}(t_i)) > 0$. Second, the j fibers of $\hat{S}$ have measure bounded below by ε'. That is, $\mu_{-j}(\hat{S}(t_j)) > \varepsilon'$.

Let $E = \{\mathbf{t} \in \mathcal{T} \mid p_i(t) > \varphi_i, t_i \in L_i\}$. Since $\hat{p}_i(t_i) > \varphi_i$ for all $t_i \in L_i \subset \mathcal{T}_i^*$, E has strictly positive measure. By taking a subset if necessary, we know that for all k, the projections E_k on $\mathcal{T}_k$ have strictly positive measure, as do the projections on $-i$ and on $-\{i,j\}$. ($E_{-i}(t_i)$ denotes, as usual, the t_i fiber of E.)

Let $A = M_i \times E_{-i}$. Since $\mu_i(M_i) > 0$ and $\mu_{-i}(E_{-i}) > 0$, we see that $\mu(A) > 0$. Taking subsets if necessary, and using Lemma 1, we know that we can find an equal measure subset (also, abusing notation, denoted A) all of whose fibers have strictly positive measure and whose projections are measurable. We now show that $p_i(\mathbf{t}) = 1$ for almost all $\mathbf{t} \in A$.

To see this, suppose not. Then we have a positive measure set of such $\mathbf{t} \in A$ with $p_i(\mathbf{t}) < 1$. For all $\mathbf{t} \in A$, we have $\hat{p}_i(t_i) > \varphi_i$. In light of Lemma 2, this implies $\sum_k p_k(\mathbf{t}) = 1$. Therefore, there exists $k \neq i$ and a positive measure set $\hat{A} \subseteq A$ such that $p_k(\mathbf{t}) > 0$ for all $\mathbf{t} \in \hat{A}$.

But fix any $\mathbf{t}' \in \hat{A}$. By construction, $t_i' \in M_i$ and $t_{-i}' \in E_{-i}(t_i'')$ for some $t_i'' \in L_i$. Since $t_i' \in M_i$ and $t_i'' \in L_i$, we have $t_i' > t_i''$, $\hat{p}_i(t_i) > \varphi_i$, and $\hat{p}_i(t_i'') > \varphi_i$. By definition of $E_{-i}(t_i'')$, we have $p_i(t_i'', \mathbf{t}_{-i}') > 0$. Finally, we have $p_k(\mathbf{t}') > 0$. Letting $\mathbf{t}'' = (t_i'', \mathbf{t}_{-i}')$, we see that this is impossible given that we removed the set $\bar{S}$ defined in Lemma 4 from $\mathcal{T}$. Hence $p_i(\mathbf{t}) = 1$ for all $\mathbf{t} \in A$.

Let $B = M_i \times \hat{S}_j \times E_{-ij}$. Recall that $\mu_i(M_i) > 0$. Also, $\mu(\hat{S}) > 0$ implies $\mu_j(\hat{S}_j) > 0$. Finally, $\mu_{-ij}(E_{-ij}) > 0$. Hence $\mu(B) > 0$. Again, taking subsets if necessary, and using Lemma 1, we know that we can find an equal measure subset (also, abusing notation, denoted B) all of whose fibers have strictly positive measure and whose projections are measurable. We now show that for all $\mathbf{t} \in B$, we have $p_j(\mathbf{t}) = 1$.

To see this, suppose not. Just as before, Lemma 2 then implies that there exists $k \neq j$ and $\hat{B} \subseteq B$ such that for all $\mathbf{t} \in \hat{B}$, $p_k(\mathbf{t}) > 0$. First, we show that $k \neq i$. To see this, suppose to the contrary that $k = i$. Fix any $\mathbf{t}'' \in \hat{B}$. By assumption, $p_i(\mathbf{t}'') > 0$. By definition of B, $t_i'' \in M_i$, so $\hat{p}_i(t_i'') > \varphi_i$. Also by definition of B, $t_j'' \in \hat{S}_j$. So fix $\mathbf{t}' \in \hat{S}$ such that $t_j' = t_j''$. By definition of $\hat{S}$, $t_i' \in U_i$, implying both $\hat{p}_i(t_i') > \varphi_i$ and $t_i' > t_i''$ (as $t_i'' \in M_i$). The definition of $\hat{S}$ also implies $p_j(\mathbf{t}') > 0$. Just as before, this contradicts the removal of $\bar{S}$ from $\mathcal{T}$. Hence $k \neq i$.

So fix any $\mathbf{t}' \in \hat{B}$. By assumption, $p_k(\mathbf{t}') > 0$. By definition of B, $t_i' \in M_i$, so $\hat{p}_i(t_i') > \varphi_i$. Also, the definition of B implies that $\mathbf{t}_{-ij}' \in E_{-ij}(t_i'')$ for some $t_i'' \in L_i$. Hence there exists t_j'' such that $(t_j'', \mathbf{t}_{-ij}') \in E_{-i}(t_i'')$. Let $\mathbf{t}'' = (t_i'', t_j'', \mathbf{t}_{-ij}')$. By construction,

$t'_k = t''_k$. Also, since $t''_i \in L_i$, we have $\hat{p}_i(t''_i) > \varphi_i$ and $t'_i > t''_i$ (as $t'_i \in M_i$). Finally, by definition of $E_{-i}(t''_i)$, we have $p_i(\mathbf{t}'') > 0$. Again, this contradicts the removal of $\bar{S}$ from $\mathcal{T}$. Hence for all $\mathbf{t} \in B$, $p_j(\mathbf{t}) = 1$.

Summarizing, for every $\mathbf{t} \in A$, we have $p_i(\mathbf{t}) = 1$ (and hence $p_j(\mathbf{t}) = 0$) and $\hat{p}_i(t_i) \geq \varphi_i + \varepsilon$, while for almost every $\mathbf{t} \in B$, we have $p_j(\mathbf{t}) = 1$ and and $\hat{p}_i(t_i) \geq \varphi_i + \varepsilon$.

For any $t'_j \in A_j$ and $t''_j \in B_j$, let

$$F_{-j}(t'_j, t''_j) = \{\mathbf{t}_{-j} \in \mathcal{T}_{-j} \mid p_j(t'_j, \mathbf{t}_{-j}) > p_j(t''_j, \mathbf{t}_{-j})\}.$$

Obviously, for every t_j and hence every $t_j \in A_j$, we have $\hat{p}_j(t_j) \geq \varphi_j$. For every $t_j \in B_j$, we have $t_j \in \hat{S}_j \subseteq S_j$, so $\hat{p}_j(t_j) = \varphi_j$. Hence for every $t'_j \in A_j$ and $t''_j \in B_j$, we have $\hat{p}_j(t'_j) \geq \hat{p}_j(t''_j)$ even though $p_j(\mathbf{t}') = 0$ and $p_j(\mathbf{t}'') = 1$ for all $\mathbf{t}' \in A$, $\mathbf{t}'' \in B$. Moreover, $B_{-j} = A_{-j} = M_i \times E_{-i,j}$. Hence for every $t'_j \in A_j$ and $t''_j \in B_j$, we must have $\mu_{-j}(F_{-j}(t'_j, t''_j)) > 0$.

By Lemma 2, the fact that $p_j(\mathbf{t}'') = 1$ for $\mathbf{t}'' \in B$ implies that $t''_j > c_j$ for all $\mathbf{t}'' \in B$. Hence, by Lemma 2, for every $(t''_j, \mathbf{t}_{-j})$ with $t''_j \in B_j$, we have $\sum_k p_k(t''_j, \mathbf{t}_{-j}) = 1$. Thus for every $t_{-j} \in F_{-j}(t'_j, t''_j)$, there exists $k \neq j$ such that $p_k(t''_j, \mathbf{t}_{-j}) > 0$.

Let

$$G = \left\{\left(t'_j, t''_j, \mathbf{t}_{-j}\right) \in \mathcal{T}_j^{(1)} \times \mathcal{T}_j^{(2)} \times \mathcal{T}_{-j} \mid t'_j \in A_j,\ t''_j \in B_j,\ \text{and } \mathbf{t}_{-j} \in F_{-j}\left(t'_j, t''_j\right)\right\}$$

where we use the superscripts on $\mathcal{T}_j$ to distinguish the order of components. The argument above implies that according to the product measure $\mu = \mu_j \times \mu_j \times \mu_{-j}$, G is non–null, i.e., $\mu(G) > 0$. (Specifically, $\mu(G) = \int_{A_j} \int_{B_j} \mu_{-j}(F_{-j}(t'_j, t''_j))\mu_j(dt'_j)\mu_j(dt''_j)$ which is strictly positive since for each (t'_j, t''_j) in the domain of integration $\mu_{-j}(F_{-j}(t'_j, t''_j)) > 0$ and the domains of integration have positive μ_j measure.) The argument above also showed that for every $(t'_j, t''_j, \mathbf{t}_{-j}) \in G$, there exists k such that $p_k(t''_j, t_{-k}) > 0$. Therefore there exists k such that $\mu(G^k) > 0$ where

$$G^k = \left\{(t'_j, t''_j, \mathbf{t}_{-j}) \in A_j \times B_j \times \mathcal{T}_{-j} \mid \mathbf{t}_{-j} \in F_{-j}(t'_j, t''_j),\ \text{and } p_k\left(t''_j, \mathbf{t}_{-j}\right) > 0\right\}.$$

So we can find $\hat{G}^k \subset G^k$ such that $\mu(\hat{G}^k) > 0$ and for all $(t'_j, t''_j, \mathbf{t}_{-j}) \in \hat{G}^k$, we have (1) $p_j(t'_j, \mathbf{t}_{-j}) > p_j(t''_j, \mathbf{t}_{-j}) + \varepsilon''$, and (2) $p_k(t''_j, \mathbf{t}_{-j}) > \varepsilon''$. Taking subsets if necessary, and using Lemma 1, we know that we can find an equal measure subset (also, abusing notation, denoted G^k) all of whose fibers have strictly positive measure and whose projections are measurable.

Now we define

$$
\begin{aligned}
\hat{C} &= \mathrm{proj}_{\mathcal{T}_j^{(2)} \times \mathcal{T}_{-j}} \hat{G}^k \\
\hat{D} &= \mathrm{proj}_{\mathcal{T}_j^{(1)} \times \mathcal{T}_{-j}} \hat{G}^k \\
\hat{A} &= A \cap \left[\mathrm{proj}_{\mathcal{T}_j^{(1)}} \hat{G}^k \times \mathcal{T}_{-j} \right] = \left\{ \mathbf{t} \in A \mid t_j \in \mathrm{proj}_{\mathcal{T}_j^{(1)}} \hat{G}^k \right\} \\
\hat{B} &= B \cap \left[\mathrm{proj}_{\mathcal{T}_j^{(2)}} \hat{G}^k \times \mathcal{T}_{-j} \right] = \left\{ \mathbf{t} \in B \mid t_j \in \mathrm{proj}_{\mathcal{T}_j^{(2)}} \hat{G}^k \right\} \\
\tilde{S} &= \hat{S} \cap \left[\mathrm{proj}_{\mathcal{T}_j^{(2)}} \hat{G}^k \times \mathcal{T}_{-j} \right] = \left\{ \mathbf{t} \in \hat{S} \mid t_j \in \mathrm{proj}_{\mathcal{T}_j^{(2)}} \hat{G}^k \right\}
\end{aligned}
$$

All the above defined sets are measurable with strictly positive measure.[2]

The following is a summary of the key facts about these sets. For every $\mathbf{t} \in \hat{A}$, we have $p_i(\mathbf{t}) = 1$ and $\hat{p}_i(t_i) \geq \varphi_i + \varepsilon$. For every $\mathbf{t} \in \tilde{S}$, we have $p_j(\mathbf{t}) \geq \varepsilon$. For every $\mathbf{t} \in \hat{C}$, we have $p_k(\mathbf{t}) \geq \varepsilon''$. For every $\mathbf{t} \in \hat{D}$, we have $p_j(\mathbf{t}) \geq \varepsilon$. Finally, $\hat{A}_j = \hat{D}_j$, $\tilde{S}_j = \hat{C}_j$, and $\hat{C}_k = \hat{D}_k$. (Also $\mu_{-j}(\hat{C}_{-j}) = \mu_{-j}(\hat{D}_{-j}) > 0$, $\mu(\hat{C}) > 0$, and $\mu(\hat{D}) > 0$. To see that $\hat{A}_j = \hat{D}_j$, note that $\hat{G}^k_{j^{(1)}} \subset A_j$. Similarly, to see that $\tilde{S}_j = \hat{C}_j$, note that $\hat{G}^k_{j^{(2)}} \subset \hat{S}_j = B_j$.)

For each $E \in \{\hat{A}, \tilde{S}, \hat{C}, \hat{D}\}$, define a function $z_E : \mathcal{T} \to [0, 1)$ such the following holds (where, for notational simplicity, the subscripts of Z do not include the hats and tildes):

$$
z_E = 0 \text{ iff } \mathbf{t} \notin E \tag{2}
$$
$$
\forall t_j \in \hat{A}_j = \hat{D}_j, \quad \mathrm{E}_{\mathbf{t}_{-j}}[z_A(t_j, \mathbf{t}_{-j})] = \mathrm{E}_{\mathbf{t}_{-j}}[z_D(t_j, \mathbf{t}_{-j})] \tag{3}
$$
$$
\forall t_k \in \hat{C}_k = \hat{D}_k, \quad \mathrm{E}_{\mathbf{t}_{-k}}[z_C(t_k, \mathbf{t}_{-k})] = \mathrm{E}_{\mathbf{t}_{-k}}[z_D(t_k, \mathbf{t}_{-k})] \tag{4}
$$
$$
\forall t_j \in \tilde{S}_j = \hat{C}_j, \quad \mathrm{E}_{\mathbf{t}_{-j}}[z_S(t_j, \mathbf{t}_{-j})] = \mathrm{E}_{\mathbf{t}_{-j}}[z_C(t_j, \mathbf{t}_{-j})] \tag{5}
$$

We show below that such functions exist. Note the following useful implication of the definitions. If we multiply both sides of the first equation by $\mu_j(t_j)$ and integrate over t_j, we obtain

$$
\mathrm{E}_{\mathbf{t}}[z_A(\mathbf{t})] = \mathrm{E}_{\mathbf{t}}[z_D(\mathbf{t})].
$$

Similarly,

$$
\mathrm{E}_{\mathbf{t}}[z_S(\mathbf{t})] = \mathrm{E}_{\mathbf{t}}[z_C(\mathbf{t})].
$$
$$
\mathrm{E}_{\mathbf{t}}[z_C(\mathbf{t})] = \mathrm{E}_{\mathbf{t}}[z_D(\mathbf{t})].
$$

---

[2]For example, $\hat{A}$ has strictly positive measure because we defined it to have fibers with strictly positive measure. Moreover, $\mathrm{proj}_{\mathcal{T}_j^{(1)}} \hat{G}^k$ is a subset of A_j with strictly positive measure. So the measure of $\hat{A}$ is the integral over a strictly positive measure set of t_j's (those in $\mathrm{proj}_{\mathcal{T}_j^{(1)}} \hat{G}^k$) of the measure of the j–fibers of A, which have strictly positive measure. The same argument applies to $\hat{B}$ and to $\tilde{S}$ (the latter since $\hat{S}_j = B_j$).

Hence

$$\mathrm{E}_{\mathbf{t}}[z_A(\mathbf{t})] = \mathrm{E}_{\mathbf{t}}[z_S(\mathbf{t})].$$

We now use this fact to construct a mechanism that improves on $\mathbf{p}$.

Define $\mathbf{p}^*$ as follows. For any $\mathbf{t} \notin \hat{A} \cup \tilde{S} \cup \hat{C} \cup \hat{D}$, $\mathbf{p}^*(\mathbf{t}) = \mathbf{p}(\mathbf{t})$. Similarly, for any $\ell \notin \{i, j, k\}$, we have $p^*_\ell(\mathbf{t}) = p_\ell(\mathbf{t})$ for all $\mathbf{t}$. Also,

$$\forall \mathbf{t} \in \hat{A}, \quad p^*_i(\mathbf{t}) = p_i(\mathbf{t}) - \varepsilon z_A(\mathbf{t}), \quad p^*_j(\mathbf{t}) = p_j(\mathbf{t}) + \varepsilon z_A(\mathbf{t}), \quad \text{and} \quad p^*_k(\mathbf{t}) = p_k(\mathbf{t})$$

$$\forall \mathbf{t} \in \tilde{S}, \quad p^*_i(\mathbf{t}) = p_i(\mathbf{t}) + \varepsilon z_S(\mathbf{t}), \quad p^*_j(\mathbf{t}) = p_j(\mathbf{t}) - \varepsilon z_S(\mathbf{t}), \quad \text{and} \quad p^*_k(\mathbf{t}) = p_k(\mathbf{t})$$

$$\forall \mathbf{t} \in \hat{C}, \quad p^*_i(\mathbf{t}) = p_i(\mathbf{t}), \quad p^*_j(\mathbf{t}) = p_j(\mathbf{t}) + \varepsilon z_C(\mathbf{t}), \quad \text{and} \quad p^*_k(\mathbf{t}) = p_k(\mathbf{t}) - \varepsilon z_C(\mathbf{t})$$

$$\forall \mathbf{t} \in \hat{D}, \quad p^*_i(\mathbf{t}) = p_i(\mathbf{t}), \quad p^*_j(\mathbf{t}) = p_j(\mathbf{t}) - \varepsilon z_D(\mathbf{t}), \quad \text{and} \quad p^*_k(\mathbf{t}) = p_k(\mathbf{t}) + \varepsilon z_D(\mathbf{t}).$$

The key facts summarized above are easily seen to imply that $p^*_\ell(\mathbf{t}) \geq 0$ for all ℓ and all $\mathbf{t}$. Also, $\sum_\ell p^*_\ell(\mathbf{t}) = \sum_\ell p_\ell(\mathbf{t})$, so the constraint that $\mathbf{p}^*$ sum to less than 1 is satisfied.

It is easy to see that the way we defined the z functions implies that $\hat{p}^*_j(t_j) = \hat{p}_j(t_j)$ for all t_j and $\hat{p}^*_k(t_k) = \hat{p}_k(t_k)$ for all t_k. Finally, note that $p^*_i(\mathbf{t}) < p_i(\mathbf{t})$ only for $t_i \in \hat{A}_i$ and that such t_i have $\hat{p}_i(t_i) \geq \varphi_i + \varepsilon$. Hence for those t_i's with $p^*_i(t_i, \mathbf{t}'_{-i}) < p_i(t_i, \mathbf{t}_{-i})$ for some $\mathbf{t}_{-i}$, we have

$$\hat{p}^*_i(t_i) \geq \hat{p}_i(t_i) - \varepsilon \mathrm{E}_{\mathbf{t}_{-i}}[z_A(\mathbf{t}_{-i}, t_i)].$$

But the fact that $z_A(\mathbf{t}) < 1$ for all $\mathbf{t}$ implies that the right–hand side is at least

$$\hat{p}_i(t_i) - \varepsilon \geq \varphi_i + \varepsilon - \varepsilon = \varphi_i.$$

Hence the constraint that $p^*_\ell(t_\ell) \geq \varphi_\ell$ holds for all t_ℓ and all ℓ. Therefore, $\mathbf{p}^*$ is feasible given φ.

Finally, note that the principal's payoff from $\mathbf{p}^*$ minus his payoff from $\mathbf{p}$ is

$$\begin{aligned}
\mathrm{E}_{t_i}\left[(\hat{p}^*_i(t_i) - \hat{p}_i(t_i))(t_i - c_i)\right] &= \varepsilon \int_{\tilde{S}} z_S(\mathbf{t})(t_i - c_i)\, \mu(d\mathbf{t}) - \varepsilon \int_{\hat{A}} z_A(\mathbf{t})(t_i - c_i)\, \mu(d\mathbf{t}) \\
&> \varepsilon(\inf U_i - c_i)\mathrm{E}[z_S(\mathbf{t})] - \varepsilon(\sup M_i - c_i)\mathrm{E}[z_A(\mathbf{t})] \\
&= \varepsilon \mathrm{E}[z_S(\mathbf{t})](\inf U_i - \sup M_i),
\end{aligned}$$

where the first inequality follows from the fact that $t_i \in \tilde{S}_i$ implies $t_i \in U_i$ and $t_i \in \hat{A}_i$ implies $t_i \in M_i$ and the last equality from $\mathrm{E}[z_S(\mathbf{t})] = \mathrm{E}[z_A(\mathbf{t})]$. Recall that $\inf U_i > \sup M_i$, so the expression above is strictly positive. Hence if such z functions exist, $\mathbf{p}$ could not have been optimal.

To conclude, we show that for each $E \in \{\hat{A}, \tilde{S}, \hat{C}, \hat{D}\}$, z_E functions exist that satisfy equations (2), (3), (4), and (5).

Fix $\delta < 1$ and define functions as follows:

$$\begin{aligned} g(t_j) &= \delta\mu_{-j}(\hat{A}_{-j}(t_j)) \\ z_A(t_j) &= \delta\int_{\hat{D}_{-j}(t_j)} \left[\mu_{-k}(\hat{C}_{-k}(t_k))\right]\mu_{-j}(d\mathbf{t}_{-j}) \\ z_C(t_k) &= \int_{\hat{D}_{-k}(t_k)} g(t_j)\mu_{-k}(d\mathbf{t}_{-k}) \\ z_D(t_k, t_j) &= g(t_j)\mu_{-k}\left(\hat{C}_{-k}(t_k)\right) \\ z_S(t_j) &= \frac{\int_{\hat{C}_{-j}(t_j)} z_C(t_k)\mu_{-j}(d\mathbf{t}_{-j})}{\mu_{-j}(\tilde{S}_{-j}(t_j))} \end{aligned}$$

where we recall that for any event S, we let $S_{-\ell}(t_\ell) = \{\mathbf{t}_{-\ell} \in \mathcal{T}_{-\ell} \mid (t_\ell, \mathbf{t}_{-\ell}) \in E\}$, the t_ℓ–fiber of E. For any $\delta < 1$, it is obvious that z_A, z_C, and z_D take values in $[0, 1)$. Regarding z_S, if $\mu(\tilde{S}(t_j))$ is bounded away from above zero, then for $\delta \leq \inf_{t_j\in\tilde{S}_j} \mu_{-j}(\tilde{S}_{-j}(t_j))$, we have $z_S \in [0, 1)$. As discussed above, $\inf_{t_j\in\tilde{S}_j} \mu_{-j}(\tilde{S}_{-j}(t_j)) > \varepsilon$ so we can find such a δ.

We now verify equations (3), (4), and (5). First, consider equation (3). Note that

$$\begin{aligned} \mathrm{E}_{\mathbf{t}_{-j}}[z_A(t_j, \mathbf{t}_{-j})] &= \int_{\hat{A}_{-j}(t_j)} z_A(t_j)\mu_{-j}(d\mathbf{t}_{-j}) \\ &= z_A(t_j)\mu_{-j}(\hat{A}_{-j}(t_j)) \\ &= \delta\mu_{-j}(\hat{A}_{-j}(t_j))\int_{\hat{D}_{-j}(t_j)} \mu_{-k}(\hat{C}_{-k}(t_k))\mu_{-j}(d\mathbf{t}_{-j}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}_{\mathbf{t}_{-j}}[z_D(t_j, \mathbf{t}_{-j})] &= \int_{\hat{D}_{-j}(t_j)} g(t_j)\mu_{-k}(\hat{C}_{-k}(t_k))\mu_{-j}(d\mathbf{t}_{-j}) \\ &= \delta\mu_{-j}(\hat{A}_{-j}(t_j))\int_{\hat{D}_{-j}(t_j)} \mu_{-k}(\hat{C}_{-k}(t_k))\mu_{-j}(d\mathbf{t}_{-j}), \end{aligned}$$

where in both sets of equalities the main step is taking terms outside the integral when they do not depend on the variable of integration. Thus (3) holds.

Second, consider equation (4). Note that

$$\begin{aligned} \mathrm{E}_{\mathbf{t}_{-k}}[z_C(t_k, \mathbf{t}_{-k})] &= z_C(t_k)\int_{\hat{C}_{-k}(t_k)} \mu_{-k}(d\mathbf{t}_{-k}) \\ &= \left[\int_{\hat{D}_{-k}(t_k)} g(t_j)\mu_{-k}(d\mathbf{t}_{-k})\right]\left[\mu_{-k}(\hat{C}_{-k}(t_k))\right] \end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}_{\mathbf{t}_{-k}}[z_D(t_k,\mathbf{t}_{-k})] &= \int_{\hat{D}_{-k}(t_k)} z_D(t_k,\mathbf{t}_{-k})\mu_{-k}(d\mathbf{t}_{-k}) \\
&= \int_{\hat{D}_{-k}(t_k)} g(t_j)\mu_{-k}(\hat{C}_{-k}(t_k))\mu_{-k}(d\mathbf{t}_{-k}) \\
&= \mu_{-k}(\hat{C}_{-k}(t_k)) \int_{\hat{D}_{-k}(t_k)} g(t_j)\mu_{-k}(d\mathbf{t}_{-k}).
\end{aligned}$$

Thus (4) holds.

Finally, consider equation (5). We have

$$\mathrm{E}_{\mathbf{t}_{-j}}[z_C(t_j,\mathbf{t}_{-j})] = \int_{\hat{C}_{-j}(t_j)} z_C(t_k)\mu_{-j}(d\mathbf{t}_{-j})$$

and

$$\begin{aligned}
\mathrm{E}_{\mathbf{t}_{-j}}[z_S(t_j,\mathbf{t}_{-j})] &= \int_{\tilde{S}_{-j}(t_j)} z_S(t_j)\mu_{-j}(d\mathbf{t}_{-j}) = z_S(t_j)\int_{\tilde{S}_{-j}(t_j)} \mu_{-j}(d\mathbf{t}_{-j}) \\
&= \frac{\int_{\hat{C}_{-j}(t_j)} z_C(t_k)\mu_{-j}(d\mathbf{t}_{-j})}{\mu_{-j}(\tilde{S}_{-j}(t_j))} \int_{\tilde{S}_{-j}(t_j)} \mu_{-j}(d\mathbf{t}_{-j}) \\
&= \int_{\hat{C}_{-j}(t_j)} z_C(t_k)\mu_{-j}(d\mathbf{t}_{-j}).
\end{aligned}$$

Thus (5) holds. ∎

Lemma 6. *For any i,*

$$\mu_i\left(\{t_i \in \mathcal{T}_i \mid \hat{p}_i(t_i) = \varphi_i\}\right) > 0.$$

Proof. Clearly if $\varphi_i = 1$, the result holds, so assume $\varphi_i < 1$.

Suppose the claim is false. Recall that the principal's objective function is

$$\sum_i \{\mathrm{E}_{t_i}[\hat{p}_i(t_i)(t_i - c_i)] + \varphi_i c_i\}$$

and that at the optimal solution $\varphi_i = \inf_{t_i} \hat{p}_i(t_i)$.

If $\mu_i(\{t_i \mid \hat{p}_i(t_i) = \varphi_i\}) = 0$, then for any $\delta > 0$, there is an $\varepsilon > 0$ such that

$$\mu_i\left(\{t_i \mid \hat{p}_i(t_i) < \varphi_i + \varepsilon\}\right) < \delta.$$

To see this, fix a sequence ε_n converging to 0 and define

$$A_n = \{t_i \mid \hat{p}_i(t_i) < \varphi_i + \varepsilon_n\},$$

$$A_0 = \{t_i \mid \hat{p}_i(t_i) = \varphi_i\},$$

and let $\delta_n = \mu_i(A_n)$. Then $A_n \downarrow A_0$ and $\mu_i(A_0) = 0$ by assumption, so $\delta_n \downarrow 0$. Hence for any $\delta > 0$, find n such that $\delta_n < \delta$ and choose $\varepsilon = \varepsilon_n$ to get the desired property.

So given any $\delta \in (0,1)$ and the corresponding ε, let $A_i^{\delta,\varepsilon} = \{t_i \mid \hat{p}_i(t_i) < \varphi_i + \varepsilon\}$. Choose δ small enough so that $\varphi_i + \varepsilon < 1 - I\sqrt{\delta}$. (This is possible since $\varphi_i < 1$.) So for each $t_i \in A_i^{\delta,\varepsilon}$, we have

$$\int_{\mathcal{T}_{-i}} p_i(t_i, \mathbf{t}_{-i})\mu_{-i}(d\mathbf{t}_{-i}) < 1 - I\sqrt{\delta}.$$

By hypothesis, $\hat{p}_i(t_i) > \varphi_i$ with probability 1. Hence by Lemma 2, we have $\sum_k p_k(\mathbf{t}) = 1$ with probability 1. Therefore, for each t_i with $\hat{p}_i(t_i) < \varphi_i + \varepsilon$, there exists $k = k^{t_i,\delta,\varepsilon} \neq i$ and $V_k^{t_i,\delta,\varepsilon} \subseteq \mathcal{T}_{-i}$ with $p_k(t_i, \mathbf{t}_{-i}) \geq \sqrt{\delta}$ for all $\mathbf{t}_{-i} \in V_k^{t_i,\delta,\varepsilon}$ and $\mu_{-i}(V_k^{t_i,\delta,\varepsilon}) \geq \sqrt{\delta}$. Choose a subset of $V_k^{t_i,\delta,\varepsilon}$ with measure $\sqrt{\delta}$ and for simplicity denote it by $V_k^{t_i,\delta,\varepsilon}$.

Let $\eta = \min\{\sqrt{\delta}, \varepsilon\}$. Increase φ_i by $\eta\sqrt{\delta}$. This change increases the value of the objective function by $c_i\eta\sqrt{\delta}$. However, this may violate the constraint that $\hat{p}_i(t_i) \geq \varphi_i$ for all t_i. Clearly, this can only occur for t_i such that $\hat{p}_i(t_i) < \varphi_i + \eta\sqrt{\delta}$. By our choice of η, such t_i satisfy $\hat{p}_i(t_i) < \varphi_i + \varepsilon\sqrt{\delta} < \varphi_i + \varepsilon$ as $\delta < 1$. So for all t_i such that $\hat{p}_i(t_i) < \varphi_i + \varepsilon$ and all $\mathbf{t}_{-i} \in V_k^{t_i,\delta,\varepsilon}$, increase $p_i(t_i, \mathbf{t}_{-i})$ by η and decrease $p_k(t_i, \mathbf{t}_{-i})$ by η. Since $\mu_{-i}(V_k^{t_i,\delta,\varepsilon}) = \sqrt{\delta}$, this change increases $\hat{p}_i(t_i)$ by $\eta\sqrt{\delta}$. Hence we again have $\hat{p}_i(t_i) \geq \varphi_i$ for all t_i after the change.

However, the reduction in p_k may have violated the constraint $\hat{p}_k(t_k) \geq \varphi_k$ for all t_k. Hence we increase φ_k by $\eta\delta$. To see that this will ensure the constraint is satisfied, note that p_k was reduced only for t_i such that $\hat{p}_i(t_i) < \varphi_i + \varepsilon$, a set with probability less than δ. Hence for any t_k, the reduction in $\hat{p}_k(t_k)$ must be less than $\eta\delta$. After this change, the resulting p and φ tuples satisfy feasibility.

To see that the objective function has increased as a result, recall that the gain from the increase in φ_i is $c_i\eta\sqrt{\delta}$. Similar reasoning shows that the loss from decreasing φ_k is $c_k\eta\delta$. Finally, the reduction in p_k and the corresponding increase in p_i generates a loss of no more than $\eta\delta\sqrt{\delta}[(\bar{t}_k - c_k) - (\underline{t}_i - c_i)]$ since the measure of the set of $\mathbf{t}$'s for which we make this change is less than $\delta\sqrt{\delta}$. Hence the objective function increases if

$$c_i\eta\sqrt{\delta} > c_k\eta\delta + \eta\delta\sqrt{\delta}\left[(\bar{t}_k - c_k) - (\underline{t}_i - c_i)\right],$$

which must hold for δ sufficiently small. ▌

Recall that

$$\mathcal{T}_i^* = \left\{t_i \in \mathcal{T}_i \setminus \bar{S}_i \mid \hat{p}_i(t_i) > \varphi_i \text{ and } \mu_i(\{t_i' \mid \hat{p}_i(t_i') > \varphi_i \text{ and } t_i' < t_i\}) > 0\right\}.$$

Lemma 7. *There exists v^* such that for all i,*

$$\mathcal{T}_i^* = \{t_i \in \mathcal{T}_i \mid t_i - c_i > v^*\}$$

up to sets of measure zero.

Proof. First, we show that for every i and j, we have $\mu_{ij}(E_{ij}) = 0$ where

$$E_{ij} = \left\{(t_i, t_j) \mid t_i - c_i > t_j - c_j,\ \hat{p}_i(t_i) = \varphi_i,\ \text{and } t_j \in \mathcal{T}_j^*\right\}.$$

To see this, suppose to the contrary that $\mu_{ij}(E_{-ij}) > 0$. Clearly, this implies $\mathcal{T}_j^* \neq \emptyset$. Let

$$F_{-ij} = \prod_{k \neq i,j} \{t_k \in \mathcal{T}_k \mid \hat{p}_k(t_k) = \varphi_k\}$$

and let $S = E_{ij} \times F_{-ij}$. Then $\mu(S) > 0$ by Lemma 6.

By Lemma 5, the fact that $t_j \in \mathcal{T}_j^*$ and that $\hat{p}_k(t_k) = \varphi_k$ for all $k \neq j$ implies that up to sets of measure zero, we must have $p_k(\mathbf{t}) = 0$ for all $k \neq j$. However, by Lemma 3, the fact that $t_i - c_i > t_j - c_j$ and $\hat{p}_j(t_j) > \varphi_j$ implies that up to sets of measure zero, we have $p_j(\mathbf{t}) = 0$. So $\sum_k p_k(\mathbf{t}) = 0$ for almost all $\mathbf{t} \in E \times F$, contradicting Lemma 2.

We now show that this implies that for all i and j such that $\mathcal{T}_i^* \neq \emptyset$ and $\mathcal{T}_j^* \neq \emptyset$, we have

$$\inf \mathcal{T}_i^* - c_i = \inf \mathcal{T}_j^* - c_j.$$

Without loss of generality, assume $\inf \mathcal{T}_i^* - c_i \geq \inf \mathcal{T}_j^* - c_j$. Suppose that there is a positive measure set of $t_i \in \mathcal{T}_i$ such that $t_i > \inf \mathcal{T}_i^*$ but $t_i \notin \mathcal{T}_i^*$. Hence for each such t_i, we must have $\hat{p}_i(t_i) = \varphi_i$. By definition of the infimum, for every $r > \inf \mathcal{T}_j^*$, there exists $t_j \in \mathcal{T}_j^*$ such that $r > t_j \geq \inf \mathcal{T}_j^*$. By definition of $\mathcal{T}_j^*$, the measure of such t_j's must be strictly positive since $t_j \in \mathcal{T}_j^*$ implies that there is a positive measure set of $t_j' < t_j$ with $t_j' \in \mathcal{T}_j^*$. But then $\mu_{ij}(E_{ij}) > 0$, a contradiction. Hence, up to sets of measure zero, $t_i > \inf \mathcal{T}_i^*$ implies $\hat{p}_i(t_i) > \varphi_i$.

By Lemma 6, then, we must have $\inf \mathcal{T}_i^* > \underline{t}_i$. So suppose, contrary to our claim, that $\inf \mathcal{T}_i^* - c_i > \inf \mathcal{T}_j^* - c_j$. Then the set of t_i such that $\inf \mathcal{T}_i^* - c_i > t_i - c_i > \inf \mathcal{T}_j^* - c_j$ and $\hat{p}_i(t_i) = \varphi_i$ has strictly positive probability. The same reasoning as in the previous paragraph shows that $\mu_{ij}(E_{ij}) > 0$, a contradiction.

In light of this, we can specify v^* such that the claim of the lemma holds. First, if $\mathcal{T}_i^* = \emptyset$ for all i, then set $v^* \geq \max_i(\bar{t}_i - c_i)$. Obviously, the lemma holds in this case.

Otherwise, let $v^* = \inf \mathcal{T}_i^* - c_i$ for any i such that $\mathcal{T}_i^* \neq \emptyset$. From the above, we see that v^* is well–defined. Let $\mathcal{I}^N$ denote the set of i with $\mathcal{T}_i^* \neq \emptyset$ and $\mathcal{I}^E$ the set of i with $\mathcal{T}_i^* = \emptyset$. By assumption, $\mathcal{I}^N \neq \emptyset$.

First, we show that for this specification of v^*, the claim of the lemma holds for all $i \in \mathcal{I}^E$. To see this, suppose to the contrary that for some $i \in \mathcal{I}^E$, we have $\bar{t}_i - c_i > v^*$. Then there is a positive measure set of $\mathbf{t}$ such that $t_j \in \mathcal{T}_j^*$ for all $j \in \mathcal{I}^N$ and $t_i - c_i > t_j - c_j$ for all $j \in \mathcal{I}^N$ and some $i \in \mathcal{I}^E$. Then Lemma 3 implies $p_j = 0$ for all $j \in \mathcal{I}^N$, Lemma 5 implies $p_i = 0$ for all $i \in \mathcal{I}^E$, and Lemma 2 implies $\sum_i p_i(\mathbf{t}) = 1$, a contradiction. Hence for all $i \in \mathcal{I}^E$, we have $v^* \geq \bar{t}_i - c_i$.

To complete the proof, we show that the claim holds for all $i \in \mathcal{I}^N$. Fix any $i \in \mathcal{I}^N$. Obviously, up to sets of measure zero, $t_i \in \mathcal{T}_i^*$ implies $t_i - c_i > \inf \mathcal{T}_i^* - c_i$, so

$$\mathcal{T}_i^* \subseteq \{t_i \in \mathcal{T}_i \mid t_i - c_i > v^*\}.$$

To prove the converse, suppose to the contrary that there is a positive measure set of t_i such that $t_i - c_i > v^*$ and $t_i \notin \mathcal{T}_i^*$. Hence there must be a positive measure set of t_i such that $t_i > \inf \mathcal{T}_i^*$ and $\hat{p}_i(t_i) = \varphi_i$. To see why, recall that $v^* = \inf \mathcal{T}_i^* - c_i$, so $t_i - c_i > v^*$ is equivalent to $t_i > \inf \mathcal{T}_i^*$. Also, $\mathcal{T}_i^*$ is the set of points that have $\hat{p}_i(t_i) > \varphi_i$ and a positive measure of smaller points also satisfying this. So if $t_i \notin \mathcal{T}_i^*$ but does have $\hat{p}_i(\hat{t}_i) > \varphi_i$, it must be that the set of smaller points satisfying this has zero measure. Hence there is a zero measure of such t_i. Hence if there's a positive measure set of points outside $\mathcal{T}_i^*$, a positive measure of them have $\hat{p}_i(t_i) = \varphi_i$. Let $\hat{\mathcal{T}}_i$ denote this set.

If there is some $j \neq i$ with $\mathcal{T}_j^* \neq \emptyset$, the same argument as above implies that $\mu(E_{ij}) > 0$, a contradiction. Hence we must have $\mathcal{T}_j^* = \emptyset$ for all $j \neq i$. Hence $\hat{p}_j(t_j) = \varphi_j$ with probability 1 for all $j \neq i$. Hence Lemma 5 implies that for all $t_i \in \mathcal{T}_i^*$, we have $p_j(\mathbf{t}) = 0$ for $j \neq i$ for almost all t_{-i}. By Lemma 2, then $p_i(\mathbf{t}) = 1$ for all $t_i \in \mathcal{T}_i^*$ and almost all $\mathbf{t}_{-i}$.

By definition, for $t_i \in \hat{\mathcal{T}}_i$, we have $\hat{p}_i(t_i) = \varphi_i < 1$.[3] Note that $t_i \in \hat{\mathcal{T}}_i$ implies that t_i is larger than some $t_i' \in \mathcal{T}_i^*$. Since $t_i' \in \mathcal{T}_i^*$ implies $\hat{p}_i(t_i') > \varphi_i$, Lemma 2 implies $t_i' > c_i$ and hence $t_i > c_i$. Hence Lemma 2 implies that for almost every $t_i \in \hat{\mathcal{T}}_i$ and almost every t_{-i}, we have $\sum_j p_j(\mathbf{t}) = 1$.

This implies that for every $t_i \in \hat{\mathcal{T}}_i$, there exists $\hat{\mathcal{T}}_{-i}(t_i) \subseteq \mathcal{T}_{-i}$ and $j \neq i$, such that $p_j(t_i, \mathbf{t}_{-i}) \geq (1 - \varphi_i)/(I - 1)$ for all $\mathbf{t}_{-i} \in \hat{\mathcal{T}}_{-i}(t_i)$. To see this, suppose not. Then there is some $t_i \in \hat{\mathcal{T}}_i$ such that for every $\mathbf{t}_{-i}$ we have $p_j(t_i, \mathbf{t}_{-i}) < (1 - \varphi_i)(I - 1)$. But then $\sum_{j \neq i} p_j(t_i, \mathbf{t}_{-i}) < 1 - \varphi_i$. Recall that $\sum_j p_j(\mathbf{t}) = 1$ for all $t_i \in \hat{\mathcal{T}}_i$ and all t_{-i}. Hence $p_i(t_i, \mathbf{t}_{-i}) > \varphi_i$ for all t_{-i}, so $\hat{p}_i(t_i) > \varphi_i$, contradicting $t_i \in \hat{\mathcal{T}}_i$. Since I is finite, this implies that there exists $j \neq i$, a positive measure subset of $\hat{\mathcal{T}}_i$, say $\hat{\mathcal{T}}_i'$, and a positive measure subset of $\mathcal{T}_{-i}$, say $\hat{\mathcal{T}}_{-i}'$, such that for every $\mathbf{t} \in \hat{\mathcal{T}}_i' \times \hat{\mathcal{T}}_{-i}'$, we have $p_j(\mathbf{t}) \geq (1 - \varphi_i)/(I - 1)$.

Fix any $t_i' \in \hat{\mathcal{T}}_i'$ such that $\mu_i(\{t_i \in \hat{\mathcal{T}}_i' \mid t_i > t_i'\}) > 0$. It is easy to see that such t_i' must exist. Since $t_i' > \inf \mathcal{T}_i^*$, it must also be true that $\mu_i(\{t_i \in \mathcal{T}_i^* \mid t_i < t_i'\}) > 0$. Given

[3] If $\varphi_i = 1$, then $\mathcal{T}_i^* = \emptyset$ which contradicts our assumption.

this, for any sufficiently small $\varepsilon > 0$, we have

$$\mu_i\left(\{t_i \in \hat{\mathcal{T}}_i' \mid t_i \geq t_i' + \varepsilon\}\right) > 0$$

$$\mu_i\left(\{t_i \in \mathcal{T}_i^* \mid t_i \leq t_i' - \varepsilon\}\right) > 0.$$

Choose any such $\varepsilon \in (0, (1-\varphi_i)/(I-1))$.

Taking subsets if necessary, then, we obtain two sets, $S^1 \subseteq \hat{\mathcal{T}}_i'$ and $S^2 \subseteq \mathcal{T}_i^*$ satisfying the following. First, $\mu_i(S^1) = \mu_i(S^2) > 0$. Second, $t_i \in S^1$ implies $t_i \geq t_i' + \varepsilon$ and $t_i \in S^2$ implies $t_i \leq t_i' - \varepsilon$.

Define $\mathbf{p}^*$ as follows. For any $\mathbf{t} \notin (S^1 \cup S^2) \times \hat{\mathcal{T}}_{-i}'$, $\mathbf{p}^*(\mathbf{t}) = \mathbf{p}(\mathbf{t})$. For any $k \neq i, j$, $p_k^*(\mathbf{t}) = p_k(\mathbf{t})$ for all $\mathbf{t}$. For $\mathbf{t} \in S^1 \times \hat{\mathcal{T}}_{-i}'$,

$$p_j^*(\mathbf{t}) = p_j(\mathbf{t}) - \varepsilon \quad \text{and} \quad p_i^*(\mathbf{t}) = p_i(\mathbf{t}) + \varepsilon.$$

For $\mathbf{t} \in S^2 \times \hat{\mathcal{T}}_{-i}'$,

$$p_j^*(\mathbf{t}) = \varepsilon \quad \text{and} \quad p_i^*(\mathbf{t}) = 1 - \varepsilon.$$

Recall that $S^2 \subseteq \mathcal{T}_i^*$ and that $p_i(\mathbf{t}) = 1$ for almost all $t_i \in \mathcal{T}_i^*$ and $t_{-i} \in \mathcal{T}_{-i}$. Hence this is equivalent to $p_j^*(\mathbf{t}) = p_j(\mathbf{t}) + \varepsilon$ and $p_i^*(\mathbf{t}) = p_i(\mathbf{t}) - \varepsilon$. Recall that $\varepsilon < (1-\varphi_i)/(I-1) \leq p_j(\mathbf{t})$ for all $\mathbf{t} \in S^1 \times \hat{\mathcal{T}}_{-i}'$ and that $\varepsilon < 1$, so we have $p_k^*(\mathbf{t}) \geq 0$ for all k and $\mathbf{t}$. Also, $\sum_k p_k^*(\mathbf{t}) = \sum_k p_k(\mathbf{t})$, so the constraint that the p_k's sum to less than one is satisfied. For any $k \neq i, j$, we have $\hat{p}_k^*(t_k) = \hat{p}_k(t_k)$ for all k and t_k so for such k, the constraint that $\hat{p}_k^*(t_k) \geq \varphi_k$ obviously holds.

For any t_j, either $\hat{p}_j^*(t_j) = \hat{p}_j(t_j)$ or

$$\hat{p}_j^*(t_j) = \hat{p}_j(t_j) - \varepsilon\mu_{-j}(S^1 \times \hat{\mathcal{T}}_{-ij}') + \varepsilon\mu_{-j}(S^2 \times \hat{\mathcal{T}}_{-ij}'),$$

where $\hat{\mathcal{T}}_{-ij}'$ is the projection of $\hat{\mathcal{T}}_{-i}'$ on $\mathcal{T}_{-ij}$. But $\mu_i(S^1) = \mu_i(S^2)$, implying $\hat{p}_j^*(t_j) = \hat{p}_j(t_j) \geq \varphi_j$ for all t_j.

For any t_i, either $\hat{p}_i^*(t_i) \geq \hat{p}_i(t_i)$ or

$$\hat{p}_i^*(t_i) = 1 - \varepsilon\mu_{-i}(\hat{\mathcal{T}}_{-i}') > 1 - \varepsilon.$$

By construction, $\varepsilon < (1-\varphi_i)/(I-1) \leq 1 - \varphi_i$. Hence $1 - \varepsilon > \varphi_i$. Hence we have $\hat{p}_i^*(t_i) \geq \varphi_i$ for all t_i. So $\mathbf{p}^*$ is feasible given φ.

Finally, the change in the principal's payoff in moving from $\mathbf{p}$ to $\mathbf{p}^*$ is

$$\mu(S^1)\varepsilon\left[\mathrm{E}(t_i - c_i \mid t_i \in S^1) - \mathrm{E}(t_i - c_i \mid t_i \in S^2)\right] \geq 2\mu(S^1)\varepsilon^2 > 0.$$

Hence $\mathbf{p}$ was not optimal, a contradiction. ▌

To see that this proves Theorem 4, let v^* be the threshold. By Lemma 7, if some i has $t_i - c_i > v^*$, then that i satisfies $\hat{p}_i(t_i) > \varphi_i$. By Lemma 3, if there is more than one such i, then only the i with the largest value (i.e., $t_i - c_i$) has a positive probability of getting the good. By Lemma 5, no j with $t_j - c_j < v^*$ has any probability of getting the good. Since $\hat{p}_i(t_i) > \varphi_i$, Lemma 2 implies that we must have $\sum_j p_j(\mathbf{t}) = 1$. Hence if some i has $t_i - c_i > v^*$, the i with the largest such value gets the good with probability 1. If any i has $t_i - c_i < v^*$, then Lemma 7 implies that $\hat{p}_i(t_i) = \varphi_i$. Thus we have a threshold mechanism. ▌

3 Proof of Theorem 3

For this proof, it is useful to give an alternative definition of t_i^*. Note that we can rearrange the definition in equation (1) as

$$\int_{\underline{t}_i}^{t_i^*} t_i f_i(t_i)\, dt_i = t_i^* F_i(t_i^*) - c_i$$

or

$$t_i^* = \mathrm{E}[t_i \mid t_i \leq t_i^*] + \frac{c_i}{F_i(t_i^*)}. \tag{6}$$

For notational convenience, number the agents so that 1 is any i with $t_i^* - c_i = \max_j(t_j^* - c_j)$ and let 2 denote any other agent so $t_1^* - c_1 \geq t_2^* - c_2$. First, we show that the principal must weakly prefer having 1 as the favored agent at a threshold of $t_2^* - c_2$ to having 2 as the favored agent at this threshold. If $t_1^* - c_1 = t_2^* - c_2$, this argument implies that the principal is indifferent between having 1 and 2 as the favored agents, so we then turn to the case where $t_1^* - c_1 > t_2^* - c_2$ and show that it must always be the case that the principal strictly prefers having 1 as the favored agent at threshold $t_1^* - c_1$ to favoring 2 with threshold $t_2^* - c_2$, establishing the claim.

So first let us show that it is weakly better to favor 1 at threshold $t_2^* - c_2$ than to favor 2 at the same threshold. First, note that if any agent other than 1 or 2 reports a value above $t_2^* - c_2$, the designation of the favored agent is irrelevant since the good will be assigned to the agent with the highest reported value and this report will be checked. Hence we may as well condition on the event that all agents other than 1 and 2 report values below $t_2^* - c_2$. If this event has zero probability, we are done, so we may as well assume this probability is strictly positive. Similarly, if both agents 1 and 2 report values above $t_2^* - c_2$, the object will go to whichever reports a higher value and the report will be checked, so again the designation of the favored agent is irrelevant. Hence we can focus on situations where at most one of these two agents reports a value above $t_2^* - c_2$ and, again, we may as well assume the probability of this event is strictly positive.

If both agents 1 and 2 report values below $t_2^* - c_2$, then no one is checked under either mechanism. In this case, the good goes to the agent who is favored under the mechanism. So suppose 1's reported value is above $t_2^* - c_2$ and 2's is below. If 1 is the favored agent, he gets the good without being checked, while he receives the good with a check if 2 were favored. The case where 2's reported value is above $t_2^* - c_2$ and 1's is below is symmetric. For brevity, let $\hat{t}_1 = t_2^* - c_2 + c_1$. Note that 1's report is below the threshold iff $t_1 - c_1 < t_2^* - c_2$ or, equivalently, $t_1 < \hat{t}_1$. Given the reasoning above, we see that under threshold $t_2^* - c_2$, it is weakly better to have 1 as the favored agent if

$$\begin{aligned} F_1(\hat{t}_1)F_2(t_2^*)\mathrm{E}[t_1 \mid t_1 \leq \hat{t}_1] &+ [1 - F_1(\hat{t}_1)]F_2(t_2^*)\mathrm{E}[t_1 \mid t_1 > \hat{t}_1] \\ &+ F_1(\hat{t}_1)[1 - F_2(t_2^*)]\left\{\mathrm{E}[t_2 \mid t_2 > t_2^*] - c_2\right\} \\ &\geq F_1(\hat{t}_1)F_2(t_2^*)\mathrm{E}[t_2 \mid t_2 \leq t_2^*] + [1 - F_1(\hat{t}_1)]F_2(t_2^*)\left\{\mathrm{E}[t_1 \mid t_1 > \hat{t}_1] - c_1\right\} \\ &+ F_1(\hat{t}_1)[1 - F_2(t_2^*)]\mathrm{E}[t_2 \mid t_2 > t_2^*]. \end{aligned} \tag{7}$$

If $F_1(\hat{t}_1) = 0$, then this equation reduces to

$$F_2(t_2^*)\mathrm{E}[t_1 \mid t_1 > \hat{t}_1] \geq F_2(t_2^*)\left\{\mathrm{E}[t_1 \mid t_1 > \hat{t}_1] - c_1\right\},$$

which must hold. If $F_1(\hat{t}_1) > 0$, then we can rewrite the equation as

$$\mathrm{E}[t_1 \mid t_1 \leq \hat{t}_1] + \frac{c_1}{F_1(\hat{t}_1)} - c_1 \geq \mathrm{E}[t_2 \mid t_2 \leq t_2^*] + \frac{c_2}{F_2(t_2^*)} - c_2. \tag{8}$$

From equation (6), the right–hand side of equation (8) is $t_2^* - c_2$. Hence we need to show

$$\mathrm{E}[t_1 \mid t_1 \leq \hat{t}_1] + \frac{c_1}{F_1(\hat{t}_1)} - c_1 \geq t_2^* - c_2. \tag{9}$$

Recall that $t_2^* - c_2 \leq t_1^* - c_1$ or, equivalently, $\hat{t}_1 \leq t_1^*$. Hence from equation (1), we have

$$\mathrm{E}(t_1) \geq \mathrm{E}[\max\{t_1, \hat{t}_1\}] - c_1.$$

A similar rearrangement to our derivation of equation (6) yields

$$\mathrm{E}[t_1 \mid t_1 \leq \hat{t}_1] + \frac{c_1}{F_1(\hat{t}_1^*)} \geq \hat{t}_1.$$

Hence

$$\mathrm{E}[t_1 \mid t_1 \leq \hat{t}_1] + \frac{c_1}{F_1(\hat{t}_1)} - c_1 \geq \hat{t}_1 - c_1 = t_2^* - c_2 + c_1 - c_1 = t_2^* - c_2,$$

implying equation (8). Hence as asserted, it is weakly better to have 1 as the favored agent with threshold $t_2^* - c_2$ than to have 2 as the favored agent with this threshold.

Suppose that $t_1^* - c_1 = t_2^* - c_2$. In this case, an argument symmetric to the one above shows that the principal weakly prefers favoring 2 at threshold $t_1^* - c_1$ to favoring 1 at the same threshold. Hence the principal must be indifferent between favoring 1 or 2 at threshold $t_1^* - c_1 = t_2^* - c_2$.

We now turn to the case where $t_1^* - c_1 > t_2^* - c_2$. The argument above is easily adapted to show that favoring 1 at threshold $t_2^* - c_2$ is strictly better than favoring 2 at this threshold if the event that $t_j - c_j < t_2^* - c_2$ for every $j \neq 1, 2$ has strictly positive probability. To see this, note that if this event has strictly positive probability, then the claim follows iff equation (7) holds with a strict inequality. If $F_1(\hat{t}_1) = 0$, this holds iff $F_2(t_2^*)c_1 > 0$. By assumption, $c_i > 0$ for all i. Also, $\underline{t}_2 < t_2^*$, so $F_2(t_2^*) > 0$. Hence this must hold if $F_1(\hat{t}_1) = 0$. If $F_1(\hat{t}_1) > 0$, then this holds if equation (9) holds strictly. It is easy to use the argument above and $t_1^* - c_1 > t_2^* - c_2$ to show that this holds.

So if the event that $t_j - c_j < t_2^* - c_2$ for every $j \neq 1, 2$ has strictly positive probability, the principal strictly prefers having 1 as the favored agent to having 2. Suppose, then, that this event has zero probability. That is, there is some $j \neq 1, 2$ such that $t_j - c_j \geq t_2^* - c_2$ with probability 1. In this case, the principal is indifferent between having 1 as the favored agent at threshold $t_2^* - c_2$ versus favoring 2 at this threshold. However, we now show that the principal must strictly prefer favoring 1 with threshold $t_1^* - c_1$ to either option and thus strictly prefers having 1 as the favored agent.

To see this, recall from the proof of Theorem 2 that the principal strictly prefers favoring 1 at threshold $t_1^* - c_1$ to favoring him at a lower threshold v^* if there is a positive probability that $v^* < t_j - c_j < t_1^* - c_1$ for some $j \neq 1$. Thus, in particular, the principal strictly prefers favoring 1 at threshold $t_1^* - c_1$ to favoring him at $t_2^* - c_2$ if there is a $j \neq 1, 2$ such that the event $t_2^* - c_2 < t_j - c_j < t_1^* - c_1$ has strictly positive probability. By hypothesis, there is a $j \neq 1, 2$ such that $t_2^* - c_2 < t_j - c_j$ with probability 1, so we only have to establish that for this j, we have a positive probability of $t_j - c_j < t_1^* - c_1$. Recall that $\underline{t}_j - c_j < t_j^* - c_j$ by definition of t_j^*. By hypothesis, $t_j^* - c_j < t_1^* - c_1$. Hence we have $t_j - c_j < t_1^* - c_1$ with strictly positive probability, completing the proof. ∎

4 Comparative Statics Proof

In this appendix, we show the claim in the text regarding the effect of changes in the cost of checking the favored agent when $I = 2$ and $F_1 = F_2 = F$. For notational ease, let 1 be the favored agent. Then the probability 1 gets the good is

$$F(t_1^*)F(t_1^* - c_1 + c_2) + \int_{t_1^*}^{\bar{t}} F(t_1 - c_1 + c_2)f(t_1)\, dt_1.$$

Differentiating with respect to c_1 gives

$$f(t_1^*)F(t_1^* - c_1 + c_2)\,\frac{\partial t_1^*}{\partial c_1} + F(t_1^*)f(t_1^* - c_1 + c_2)\,\left[\frac{\partial t_1^*}{\partial c_1} - 1\right] \\ - F(t_1^* - c_1 + c_2)f(t_1^*)\,\frac{\partial t_1^*}{\partial c_1} - \int_{t_1^*}^{\bar{t}} f(t_1 - c_1 + c_2)f(t_1)\,dt_1$$

or

$$F(t_1^*)f(t_1^* - c_1 + c_2)\,\left[\frac{\partial t_1^*}{\partial c_1} - 1\right] - \int_{t_1^*}^{\bar{t}} f(t_1 - c_1 + c_2)f(t_1)\,dt_1.$$

Recall that t_1^* is defined by

$$\int_{\underline{t}}^{t_1^*} F(s)\,ds = c_1.$$

Using this, it's easy to see that

$$\frac{\partial t_1^*}{\partial c_1} = \frac{1}{F(t_1^*)}.$$

Substituting, the derivative is

$$f(t_1^* - c_1 + c_2)[1 - F(t_1^*)] - \int_{t_1^*}^{\bar{t}} f(t_1 - c_1 + c_2)f(t_1)\,dt_1 \\ = \int_{t_1^*}^{\bar{t}} [f(t_1^* - c_1 + c_2) - f(t_1 - c_1 + c_2)]f(t_1)\,dt_1.$$

Hence if f is increasing throughout the relevant range, this is negative, implying that the probability 1 gets the good is decreasing in c_1. If f is decreasing throughout the relevant range, this is positive, so 1's probability of getting the good increases in c_1. If the types have a uniform distribution, the derivative is zero.