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Switching costs in infinitely repeated games

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Abstract

We show that small switching costs can have surprisingly dramatic effects in infinitely repeated games if these costs are large relative to payoffs in a single period. This shows that the results in Lipman and Wang do have analogs in the case of infinitely repeated games [Lipman, B., Wang, R., 2000. Switching costs in frequently repeated games. J. Econ. Theory 93, August 2000, 149–190]. We also discuss whether the results here or those in Lipman–Wang imply a discontinuity in the equilibrium outcome correspondence with respect to small switching costs. We conclude that there is not a discontinuity with respect to switching costs but that the switching costs do create a discontinuity with respect to the length of a period.

JEL classification: C73

1. Introduction

Lipman and Wang (2000) showed that switching costs can have surprisingly strong effects in frequently but finitely repeated games. More specifically, suppose we have a finite stage game where the length of each period is Δ and the total length of time of play is equal to $\mathcal{L} = (T + 1)\Delta$ for some integer T. Suppose the payoff to player i from the sequence of action profiles $(a^0, \dots, a^T)$ is given by

$$\sum_{t=0}^{T}\left[\Delta u_i\left(a^t\right) - \varepsilon I_i\left(a^t, a^{t-1}\right)\right] \tag{1}$$

where $I_i(a, a') = 0$ if $a_i = a'_i$ and 1 otherwise.[1] In other words, he receives the payoff associated with each action vector played times the length of time these actions are played, minus a cost for each time he himself changes actions. We showed some very unexpected behavior in such games for small ε and Δ as long as ε is sufficiently large relative to Δ. For example, in games like the Prisoners' Dilemma which have a unique subgame perfect equilibrium outcome without switching costs, we obtain multiple equilibrium outcomes. In other games, such as coordination games, which

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[1] For simplicity, define $a_i^{-1} = a_i^0$ for all i.

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have multiple equilibria without switching costs, we showed that one can have a unique subgame perfect equilibrium with small switching costs.

The analysis used finite repetition in a critical way. We noted that if the switching cost is large relative to one period worth of payoff, then no player would find it worthwhile to change actions in the last period regardless of what actions were played in the preceding period. This causes the usual backward induction arguments to break down. The fact that actions must be fixed at the end can have large effects early in the game.

Here we consider the effect of switching costs in an infinitely repeated game for four reasons. First, given the way our earlier analysis exploited the finite horizon, it is not obvious whether similar effects could be obtained in an infinitely repeated game. Second, our earlier analysis had the drawback that it was impossible to give many general characterization results. A natural conjecture is that the simplicity of the infinite horizon may allow us to characterize the set of equilibrium payoffs, at least for "sufficiently patient" players.

Third, just as with our earlier paper, we seek to explore to what extent the standard analysis is robust with respect to modifications of the model which seem "small." A cost to changing actions from one period to the next seems natural for at least two reasons. First, it is a simple way of capturing a type of bounded rationality. Intuitively, it is easier to continue doing the same thing as in the past than it is to move to some new course of action. Second, in many economic settings, changing actions requires real costs. For example, entering or exiting a market involves obvious costs. Changing prices often requires printing new menus or advertising the new prices in some fashion.

A subtle question surrounds when such modifications of the standard model are "small." Most models of dynamic oligopoly ignore menu costs, evidently under the hypothesis that such small costs are irrelevant. However, while the cost of changing prices presumably is small relative to the present value of the firm's profits, these costs may be quite large relative to a day's worth of profits. One implication of our analysis is that the standard Folk Theorem does not hold when costs are large relative to one period's worth of payoff, even if they are small relative to the present value of payoffs.

Finally, consideration of infinitely repeated games is necessary to determine whether our earlier results indicate a discontinuity in the equilibrium outcome correspondence. To understand this, note that our earlier results indicated that subgame perfect equilibria with small ε and Δ are quite different from equilibria of finitely repeated games with $\varepsilon = 0$. Does this mean that the equilibrium outcome is discontinuous in ε at $\varepsilon = 0$? The difficulty in answering this question comes from the fact that our results all require ε large relative to Δ. Hence if ε goes to zero, to maintain our results, we must take Δ to zero as well. However, we wish to keep the total length of the game fixed. Hence if the length of a period goes to 0, the number of periods must go to infinity. Hence we are forced to turn to infinitely repeated games to address the question.

Here we show that different but also surprising results are possible with switching costs in infinitely repeated games if the switching cost is large relative to one period's worth of payoff. That is, just as in our earlier analysis, we consider switching costs which are small in the sense that the cost of one change of action is small relative to total game payoffs that can be earned over the entire horizon. However, in the case of primary interest, this cost is large relative to the game payoff which can be earned in a single period. As in our previous work, we parameterize the length of a period and primarily focus on the case where the length of a period and the switching cost are both small but the latter is large relative to the former.

To be more precise, consider player i's payoff to an infinite sequence of action profiles $a^0, a^1, \ldots$. Suppose that, as in the finite horizon case discussed above, actions are changed only at intervals of length Δ and the stage game payoffs are flow rates. It seems natural to view the switching cost as an immediate payment, not a flow cost. Under these assumptions, the agent's payoff to this infinite sequence of actions is

$$\sum_{t=0}^{\infty} \int_{t\Delta}^{(t+1)\Delta} \mathrm{e}^{-rs} u_i\big(a^t\big)\,\mathrm{d}s - \sum_{t=0}^{\infty} \mathrm{e}^{-rt\Delta} \varepsilon I_i\big(a^{t-1}, a^t\big)$$

where $I_i(a, a') = 0$ if $a_i = a'_i$ and 1 otherwise as before[2] and r is the (continuous time) discount rate. If we carry out the integration, normalize $r = 1$, and set $\delta = \mathrm{e}^{-\Delta}$, we get

[2] Also, as before, let $a_i^{-1} = a_i^0$.

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(a^t) - \sum_{t=0}^{\infty}\delta^t \varepsilon I_i(a^{t-1}, a^t). \tag{2}$$

Note that as the length of a period, Δ, gets small, δ approaches 1. Note that the cost of one change of action relative to one period worth of payoff is on the order of $\varepsilon/(1-\delta)$ and so becomes large as $\Delta \downarrow 0$ or $\delta \uparrow 1$. On the other hand, the cost of one change of action relative to all game payoffs earned is on the order of $\varepsilon/[(1-\delta)\sum_t \delta^t] = \varepsilon$. Hence this is not affected by δ and converges to 0 as $\varepsilon \downarrow 0$. Consequently, this formulation enables us to make the cost of switching large relative to one period worth of payoff while keeping it small relative to the whole repeated game's payoffs.

By contrast, consider instead a simple variation on the usual discounting formulation, where we evaluate paths of play by the discounted sum over periods of the payoff in a period minus a switching cost if incurred in that period. More specifically, suppose player i's payoff to a sequence of action profiles $a^0, a^1, \ldots$ is

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t \big[u_i(a^t) - \varepsilon I_i(a^{t-1}, a^t)\big]. \tag{3}$$

This formulation gives no obvious way to shrink the switching cost relative to the whole repeated game worth of payoff without shrinking it relative to the payoff in a single period. As before, period length can be thought of as affecting the discount rate δ. However, here δ affects game payoffs and switching costs in the same way. Hence if we reduce ε, we *must* reduce it relative to $u_i(a)$ and thus relative to one period worth of payoff. (As we explain below, there is a sense in which our formulation using equation (2) nests this alternative as a special case.)

We consider two different infinitely repeated games. In the first, each player i evaluates sequences of actions by the payoff criterion in Eq. (2). We denote this game $G(\varepsilon, \delta)$ where $\varepsilon \in [0, \infty)$ and $\delta \in [0, 1)$. The second infinitely repeated game we consider has game payoffs defined by the limit of means criterion instead of discounting. More precisely, we define the game $G_\infty(\varepsilon)$ to be the game where each player i evaluates sequences of actions by the payoff criterion

$$\liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} u_i(a^t) - \varepsilon \#\{t \mid a_i^t \neq a_i^{t-1}\}$$

where # denotes cardinality.[3] As we explain in more detail in Section 4, there is a natural sense in which this game is the limit of our finitely repeated game as $\Delta \downarrow 0$. Let $\mathcal{U}(\varepsilon, \delta)$ denote the set of equilibrium payoffs of $G(\varepsilon, \delta)$ and let $\mathcal{U}_\infty(\varepsilon)$ denote the set of equilibrium payoffs of $G_\infty(\varepsilon)$.

First, we consider $\mathcal{U}(\varepsilon, \delta)$. In line with the intuition suggested above, our results show that the set of equilibrium payoffs is exactly the usual Folk Theorem set if the switching cost is small relative to a period worth of payoff but differs from the usual set if the cost is large relative to one period of payoff. In other words, consider the limit of the set $\mathcal{U}(\varepsilon, \delta)$ as $(\varepsilon, \delta) \to (0, 1)$. This limit will depend on the particular (ε, δ) sequence chosen. We show that if we consider sequences such that $\varepsilon/(1-\delta) \to 0$, then the limiting set of payoffs is the same as the usual Folk Theorem set. That is, for such sequences, $\mathcal{U}(\varepsilon, \delta)$ converges to the set of feasible, individually rational payoffs. (These limits are defined more precisely in Section 2.) Note that such sequences can be thought of as including the formulation in equation (3) as a special case. If we use the payoff criterion in equation (2) but set $\varepsilon = \hat{\varepsilon}(1-\delta)$, we obtain equation (3) with $\hat{\varepsilon}$ replacing ε. If we take $(\varepsilon, \delta) \to (0, 1)$ in equation (2) in such a way that $\varepsilon/(1-\delta) \to 0$, this corresponds to taking $(\hat{\varepsilon}, \delta) \to (0, 1)$ in Eq. (3).

At the opposite extreme, if we consider a sequence such that $\varepsilon/(1-\delta)$ goes to infinity, we get a limiting set of payoffs which differs from the Folk Theorem set in two ways. First, the payoff a player can guarantee himself is smaller with switching costs. Intuitively, if a player needs to randomize to avoid punishment, the expected costs of switching actions makes this too costly. That is, we must appropriately redefine individual rationality. Second, the notion of feasibility changes as well since the switching costs can dissipate payoffs even in the limit as $\varepsilon \downarrow 0$. For example, in the coordination game

	a	b
a	3, 3	0, 0
b	0, 0	1, 1

[3] To ensure that this is well-defined, we allow $-\infty$ as a payoff. In other words, we treat payoffs in the repeated game as elements of $\mathbf{R} \cup \{-\infty\}$.

the usual Folk Theorem set is all payoff vectors (u_1, u_2) where $u_1 = u_2$ and $0.75 \leqslant u_i \leqslant 3$. By contrast, if $(\varepsilon, \delta) \to (0, 1)$ with $\varepsilon/(1-\delta) \to \infty$ along the sequence, the set of equilibrium payoffs converges to the set of all (u_1, u_2) such that $(0, 0) \leqslant (u_1, u_2) \leqslant (3, 3)$.

Of course, the requirement that $\varepsilon/(1-\delta) \to \infty$ is quite strong. We show two results regarding intermediate values of $\varepsilon/(1-\delta)$. First, we show that for any of the payoffs we obtain when ε is becoming arbitrarily large relative to $1-\delta$, we can approximately achieve this payoff while keeping $\varepsilon/(1-\delta)$ bounded. To make the approximation arbitrarily accurate requires making $\varepsilon/(1-\delta)$ arbitrarily large.

We also show that the requirement that $\varepsilon/(1-\delta)$ becomes arbitrarily large is driven entirely by the change in feasibility, not the change in individual rationality. More specifically, if the switching cost is on the order of two periods worth of payoff, then every payoff which is feasible (in the traditional sense) and individually rational (in our modified sense) is a limiting equilibrium payoff. Hence even intuitively small switching costs require us to modify the usual definition of individual rationality.

As we explain in Section 4, one way to understand these results is to note that if ε is not arbitrarily small relative to $1-\delta$, then there is a sense in which $G(\varepsilon, \delta)$ is bounded away from $G(0, \delta)$. In other words, when we consider a sequence of (ε, δ) such that

$$\lim_{\varepsilon \downarrow 0, \delta \uparrow 1} \mathcal{U}(\varepsilon, \delta) \neq \lim_{\delta \uparrow 1} \mathcal{U}(0, \delta),$$

we must also have

$$\lim_{\varepsilon \downarrow 0, \delta \uparrow 1} G(\varepsilon, \delta) \neq \lim_{\delta \uparrow 1} G(0, \delta).$$

(These limits are defined more precisely in Sections 2 and 4.) In this sense, these results do not indicate a discontinuity in the equilibrium outcome correspondence with respect to (ε, δ).

Next, we turn to the case where we use the limit of means to evaluate game payoffs. In this case, the switching cost is larger than the payoffs for *any* finite number of periods, so, naturally, we would expect the cost to have the largest effect here. In fact, Theorem 6 shows that both of the two earlier differences between the equilibrium payoff set and the usual Folk Theorem set remain and a third is added. This third difference is strikingly unusual: payoffs that are supported by putting some weight on payoff vectors that are not individually rational (in the modified sense appropriate for switching costs) cannot be obtained. For example, in the Prisoners' Dilemma,

	C	D
C	3, 3	0, 4
D	4, 0	2, 2

the usual Folk Theorem set is all feasible payoffs where each player gets at least 2. As $\varepsilon \downarrow 0$, the set of equilibrium payoffs of $G_\infty(\varepsilon)$ converges to the set of payoffs where each player gets at least 2 and neither gets more than 3. We get this result because in this game, we cannot put any "weight" on the $(4, 0)$ or $(0, 4)$ payoff vector. (Payoff vectors which are not convex combinations of $(3, 3)$ and $(2, 2)$ are obtained by players dissipating payoffs through the switching costs.) Intuitively, with this formulation, any path of play in the game must have the property that players change actions only finitely often with probability one. Hence any path eventually "absorbs" in the sense that at some point, actions never change again. It is obvious that we cannot have an equilibrium where the players know that actions will never change again from (C, D) or (D, C) since the player getting 0 will change actions. What is less obvious is why we cannot have some kind of randomization that "hides" from the players the fact that no further changes of action will occur. (We do allow the players to condition on public randomizing devices, so we give the maximum possible ability for the players to use such strategies.) We show that players must eventually become sure enough that no change will occur that they will deviate from any such proposed equilibrium.

Again, this result does not show a discontinuity in the equilibrium outcome correspondence with respect to ε. While

$$\lim_{\varepsilon \downarrow 0} \mathcal{U}_\infty(\varepsilon) \neq \mathcal{U}_\infty(0),$$

we will show in Section 4 that

$$\lim_{\varepsilon \downarrow 0} G_\infty(\varepsilon) \neq G_\infty(0).$$

Finally, we use Theorem 6 and results in Lipman–Wang (2000) to address whether our earlier results demonstrate a discontinuity in the equilibrium outcome correspondence. Let $G_f(\varepsilon, \Delta)$ be the game studied in Lipman–Wang (2000) using the payoff function (1) above and let $\mathcal{U}_f(\varepsilon, \Delta)$ denote the set of equilibrium payoffs. Analogously to the above, we show that when

$$\lim_{\varepsilon\downarrow 0, \Delta\downarrow 0} \mathcal{U}_f(\varepsilon, \Delta) \neq \lim_{\Delta\downarrow 0} \mathcal{U}_f(0, \Delta),$$

it is because

$$\lim_{\varepsilon\downarrow 0, \Delta\downarrow 0} G_f(\varepsilon, \Delta) \neq \lim_{\Delta\downarrow 0} G_f(0, \Delta).$$

Thus there is no discontinuity with respect to (ε, Δ). On the other hand, as above, there is in general a discontinuity with respect to Δ for any fixed $\varepsilon > 0$. Specifically,

$$\lim_{\Delta\downarrow 0} \mathcal{U}_f(\varepsilon, \Delta) \neq \mathcal{U}_\infty(\varepsilon)$$

even though

$$\lim_{\Delta\downarrow 0} G_f(\varepsilon, \Delta) = G_\infty(\varepsilon).$$

The possibility of such a discontinuity with $\varepsilon = 0$ is well known, but the discontinuity when $\varepsilon > 0$ is of a very different nature. The known discontinuity for $\varepsilon = 0$ is simply the difference between finitely and infinitely repeated games. However, this discontinuity is a failure of lower semicontinuity, not upper, as the limiting set of equilibrium payoffs is smaller than the set at the limit. The discontinuity with $\varepsilon > 0$ may have the limiting set larger than the set at the limit. Also, the discontinuity for $\varepsilon = 0$ occurs for a different set of games than the discontinuity for $\varepsilon > 0$.

Our results differ from those of Chakrabarti (1990) who considers a similar model. He analyzes infinitely repeated games with a more general "inertia cost" than we consider. His payoff criterion, however, does not fit into the class we consider. Specializing his switching cost to our setting, he assumes players evaluate payoffs to the sequence $(a^0, a^1, \ldots)$ by

$$\liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \left[u_i\left(a^t\right) - \varepsilon I_i\left(a^{t-1}, a^t\right)\right],$$

the limit of means analog of (3) above. His results are a special case of those obtained for stochastic games by Dutta (1995), while ours are not. More specifically, using Dutta's results, one can show that Chakrabarti's set of equilibrium payoffs differs from the usual Folk Theorem set in two ways, namely the two present in our Theorem 3. That is, both individual rationality and feasibility must be redefined to take account of the switching costs.[4] However, the third effect we obtain in Theorem 6 is not present. He does not discuss continuity issues.

In the next section, we state the model. In Section 3, we give our characterizations of equilibrium payoffs. In Section 4, we define a notion of closeness of games and use this to consider continuity of the equilibrium outcome correspondence. Proofs not in the text are contained in the Appendix.

2. Model

Fix a finite stage game $G = (A, u)$ where $A = A_1 \times \cdots \times A_I$, each A_i is finite and contains at least two elements, and where $u : A \to \mathbf{R}^I$. Let S_i denote the set of mixed stage game strategies—that is, S_i is the set of randomizations over A_i. We allow the players to use public randomizing devices, so a strategy for the repeated game can depend on the history of play as well as the outcome of the public randomization. For simplicity, we will suppose that there is an i.i.d. sequence of random variables, ξ_t, which are uniformly distributed on [0, 1] which all players observe. A strategy for player i, then, is a function from the history of past actions and the realization of the randomizations (up to and including the current period) into S_i. That is, it is a function $\sigma_i : \bigcup_{t=0}^{\infty} A^t \times [0,1]^t \to S_i$ where $A^0 \times [0,1]^0$ is defined to be the singleton set containing the "empty history" e.

[4] Chakrabarti states his results in a different but equivalent way.

Remark 1. As shown by Fudenberg and Maskin (1991), the use of public randomization is purely a matter of convenience in the usual repeated game. More specifically, one can obtain the same characterization of equilibrium payoffs without public randomizations. However, the assumption is not as innocuous here. While it is not needed for any of the other results, the result of Theorem 6 is not true in general without public randomization. The reason is that the equilibrium construction which is typically used to replace public randomization requires numerous changes of action. Since such changes of action are costly in this model, such behavior can be difficult to support as an equilibrium. On the other hand, the most interesting aspect of Theorem 6 is the payoffs which *cannot* be achieved. Since allowing public randomization can only increase the set of equilibrium payoffs, this is the most interesting case to consider for that result.

The payoffs in the game $G(\varepsilon, \delta)$, $\varepsilon \geqslant 0$, $\delta \in [0, 1)$, are defined as follows. Given a sequence of actions $(a^0, a^1, \ldots)$ where $a^t = (a_1^t, \ldots, a_I^t)$, i's payoff from this sequence is

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i\big(a^t\big) - \sum_{t=0}^{\infty}\delta^t \varepsilon I_i\big(a^{t-1}, a^t\big)$$

where $I_i(a^{t-1}, a^t) = 1$ if $a_i^{t-1} \neq a_i^t$ and 0 otherwise.[5] Let $\mathcal{U}(\varepsilon, \delta)$ denote the closure of the set of subgame perfect equilibrium payoffs in $G(\varepsilon, \delta)$.

Letting # denote cardinality, we define the payoff from this sequence in the game $G_\infty(\varepsilon)$ to be

$$\left[\liminf_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}u_i(a^t)\right] - \varepsilon\#\big\{t \mid a_i^t \neq a_i^{t-1}\big\}$$

if this is a real number and $-\infty$ otherwise. Note that the payoff is a well defined real number if i changes actions only finitely often. However, if i changes actions infinitely often, then the switching cost makes this payoff arbitrarily negative; hence we define the payoff to be $-\infty$. Let $\mathcal{U}_\infty(\varepsilon)$ denote the closure of the set of subgame perfect equilibrium payoffs in $G_\infty(\varepsilon)$.

We are interested in the set of $\mathcal{U}(\varepsilon, \delta)$ for ε very close to 0 and δ very close to 1. As we will see, this set will depend on the relationship of ε and δ. In addition, we are less interested in specific values of ε and δ than in general properties for (ε, δ) near $(0, 1)$. Consequently, it will prove most convenient to consider the set of limit points of $\mathcal{U}(\varepsilon_n, \delta_n)$ as $n \to \infty$ for various sequences $(\varepsilon_n, \delta_n)$.

In particular, for any $k \in [0, \infty]$, we define the set

$$\lim_{\varepsilon\downarrow 0, \delta\uparrow 1}{}^{k}\ \mathcal{U}(\varepsilon, \delta)$$

to be the set of $u \in \mathbf{R}^I$ such that there are sequences ε_n, δ_n, and u^n such that

$$\varepsilon_n > 0,\ \delta_n \in [0, 1), \quad \text{and} \quad u^n \in \mathcal{U}(\varepsilon_n, \delta_n), \quad \forall n,$$
$$\lim_{n\to\infty}\frac{\varepsilon_n}{1-\delta_n} = k,$$

and $(\varepsilon_n, \delta_n, u^n) \to (0, 1, u)$ as $n \to \infty$. Intuitively, then, k is a measure of how large ε is relative to $1-\delta$ along the sequence. The case of $k = 0$ is effectively the situation where we take ε to zero first and then δ to 1; the case of $k = \infty$ is analogous to the reverse order of limits.

The issue of ε relative to $1-\delta$ is irrelevant in the game $G_\infty(\varepsilon)$. Hence we simply define

$$\lim_{\varepsilon\downarrow 0}\mathcal{U}_\infty(\varepsilon)$$

to be the set of $u \in \mathbf{R}^I$ such that there exist sequences ε_n and u^n with

$$\varepsilon_n > 0 \quad \text{and} \quad u^n \in \mathcal{U}_\infty(\varepsilon_n), \quad \forall n$$

[5] We define a_i^{-1} to be equal to a_i^0 for any sequence of actions $(a^0, a^1, \ldots)$. In other words, there is no cost of "changing" actions in the first period regardless of the action played in that period.

with $(\varepsilon_n, u^n) \to (0, u)$ as $n \to \infty$.

The usual Folk Theorem sets have $\varepsilon = 0$—that is, they are $\mathcal{U}_\infty(0)$ and $\lim_{\delta\uparrow 1}\mathcal{U}(0,\delta)$. We define the latter analogously to the approach used above. That is, $\lim_{\delta\uparrow 1}\mathcal{U}(0,\delta)$ is the set of u such that there is a sequence δ_n converging to 1 from below and a sequence u^n converging to u with $u^n \in \mathcal{U}(0,\delta_n)$ for all n.

Define i's *reservation payoff*, v_i, by

$$v_i = \min_{s_{\sim i}\in S_{\sim i}}\Big[\max_{s_i\in S_i} u_i(s_i, s_{\sim i})\Big].$$

Let

$$R = \{u \in \mathbf{R}^I \mid u \geqslant v\}$$

denote the usual set of individually rational payoffs where $v = (v_1, \ldots, v_I)$.[6] In the case of two players,[7] the classic minmax theorem states that the order of the minimization and maximization don't matter. That is, in this case, v_i is also equal to $\max_{s_i\in S_i}[\min_{s_{\sim i}\in S_{\sim i}} u_i(s_i, s_{\sim i})]$. However, even in the two player case, if i is restricted to pure strategies, the order matters very much. We will see that the relevant reservation utility for i in the game with switching costs is what we will call i's *pure reservation payoff*, w_i, defined by

$$w_i = \max_{a_i\in A_i}\Big[\min_{s_{\sim i}\in S_{\sim i}} u_i(a_i, s_{\sim i})\Big].$$

Let

$$W = \{u \in \mathbf{R}^I \mid u \geqslant w\}$$

denote what we will call the set of *weakly individually rational* payoffs, where $w = (w_1, \ldots, w_I)$. Note that $w_i \leqslant v_i$ for all i so $R \subseteq W$. Any action $a_i \in A_i$ such that

$$\min_{s_{\sim i}\in S_{\sim i}} u_i(a_i, s_{\sim i}) = w_i$$

will be referred to as *a maxmin action for i*.

It is worth noting for future use, that

$$w_i = \max_{a_i\in A_i}\Big[\min_{a_{\sim i}\in A_{\sim i}} u_i(a_i, a_{\sim i})\Big].$$

To see this, simply note that for any $a_i \in A_i$ and any $j \neq i$, $u_i(a_i, s_{\sim i})$ is linear in j's mixed strategy. Hence the value of this expression when minimized over s_j is unaffected by restricting j to pure actions. We exploit this fact in what follows.

For any set $B \subseteq \mathbf{R}^I$, let $\mathrm{conv}(B)$ denote its convex hull. Let U denote the set of payoffs feasible from pure strategies and let F denote the usual set of feasible payoffs. That is,

$$U = \{u \in \mathbf{R}^I \mid u = u(a),\ \text{for some } a \in A\}$$

and $F = \mathrm{conv}(U)$. For comparison purposes, we first state the usual Folk Theorem.

We define a game to be *regular* if there is $u = (u_1, \ldots, u_I) \in F \cap R$ such that $u_i > v_i$ for all i.

Theorem 1 *(The Folk Theorem). For any regular game,*

A. $\mathcal{U}_\infty(0) = F \cap R$.
B. *If, in addition, F has dimension I,*

$$\lim_{\delta\uparrow 1}\mathcal{U}(0,\delta) = F \cap R.$$

This result is a trivial extension of theorems in Fudenberg and Tirole (1991, Chapter 5), and so we omit the proof.

[6] Given vectors x and y, we use $x \geqslant y$ to mean greater than or equal to in every component and $x \gg y$ to mean strictly larger in every component.

[7] Or if we allow the players other than i to use correlated strategies.

Remark 2. The restriction to regular games in Theorem 1 is generally not stated but is often used in some form. For example, one typical version of the Folk Theorem is that the equilibrium payoff set includes all feasible payoffs where each player i receives strictly more than v_i. We use the assumption to be able to give an exact statement of equilibrium payoff sets without having to consider tedious boundary calculations. The additional assumption we use in Theorem 1.B is the most simply stated sufficient condition. It could be replaced by the weaker NEU condition of Abreu et al. (1994).

3. Results

First, we consider the case of discounting. Recall that

$$\lim_{\varepsilon\downarrow 0,\delta\uparrow 1}{}^{k}\ \mathcal{U}(\varepsilon,\delta)$$

is the set of limiting equilibrium payoffs in $G(\varepsilon,\delta)$ for sequences (ε,δ) converging to $(0,1)$ such that $\varepsilon/(1-\delta)\to k$. As explained in the introduction, if the switching cost is small relative to one period worth of payoff, we expect to obtain the same results as in the usual analysis. That is, we expect the $\lim^k$ set to equal the usual Folk Theorem set if k is small. This intuition is confirmed by

Theorem 2. *For any regular game such that F has dimension I,*

$$\lim_{\varepsilon\downarrow 0,\delta\uparrow 1}{}^{0}\ \mathcal{U}(\varepsilon_n,\delta_n)=F\cap R.$$

That is, if $\varepsilon/(1-\delta)$ goes to 0 along the sequence, the limiting payoff set is the set of feasible, individually rational payoffs.

To see that the limiting set of payoffs is contained in $F\cap R$, suppose not. First, suppose that $u\in\lim^0_{\varepsilon\downarrow 0,\delta\uparrow 1}\mathcal{U}(\varepsilon_n,\delta_n)$, but $u\notin R$. Let $(\varepsilon_n,\delta_n,u^n)$ be the required sequence converging to $(0,1,u)$ for which $u^n\in\mathcal{U}(\varepsilon_n,\delta_n)$ and $\varepsilon_n/(1-\delta_n)\to 0$. Fix any i for whom $u_i<v_i$. Since $\varepsilon_n/(1-\delta_n)\to 0$ and $u_i^n\to u_i$, it must be true that

$$u_i^n<v_i-\frac{\varepsilon_n}{1-\delta_n}$$

for all n sufficiently large. But then even if it requires changing actions every period, i can switch to the strategy of choosing a myopic best reply in every period to the strategies of the opponents for that period and be better off, a contradiction.

Next, suppose $u\in\lim^0_{\varepsilon\downarrow 0,\delta\uparrow 1}\mathcal{U}(\varepsilon_n,\delta_n)$, but $u\notin F$. Again, fix the required sequence $(\varepsilon_n,\delta_n,u^n)$. Since $u^n\in\mathcal{U}(\varepsilon_n,\delta_n)$, there is some probability distribution, say q_n over A such that

$$u_i^n=\sum_{a\in A}q_n(a)u_i(a)-X_n$$

where X_n is the expected discounted switching costs in the equilibrium. We know that X_n must be bounded above by the cost of switching actions in every period so $X_n\leqslant\varepsilon_n/(1-\delta_n)$. Hence $X_n\to 0$ as $n\to\infty$. So

$$u=\lim_{n\to\infty}u^n=\lim_{n\to\infty}\sum_{a\in A}q_n(a)u_i(a).$$

Hence $u\in F$.

So we see that

$$\lim_{\varepsilon\downarrow 0,\delta\uparrow 1}{}^{0}\ \mathcal{U}(\varepsilon_n,\delta_n)\subseteq F\ \cap\ R.$$

The proof that every payoff in $F\cap R$ is a limiting equilibrium payoff is a simple extension of arguments in Fudenberg and Tirole and so is omitted.[8]

[8] The argument is to note that their proof for the observable mixed strategy case involves *strict* payoff comparisons. Hence small enough switching costs cannot affect the optimality of the strategies in question. Since the public randomization effectively creates observable mixed strategies, this completes the argument.

As discussed in the introduction, we expect differences from the standard model when the switching cost is large relative to one period's worth of payoff. That is, we expect $\lim^{k}_{\varepsilon\downarrow 0,\delta\uparrow 1}\mathcal{U}(\varepsilon,\delta)$ to differ from the usual Folk Theorem set when k is sufficiently large. This intuition is confirmed by the next result.

First, we require a few definitions. Given a set $B \subseteq \mathbf{R}^I$, let $c(B)$ denote the comprehensive, convex hull of B. That is, $c(B)$ is the set of points less than or equal to a convex combination of points in B. Define $F^* = c(U)$. This is the feasible set of payoffs when we allow players the ability to "throw away" utility.

Define a game to be *weakly regular* if there is a payoff vector $u = (u_1, \ldots, u_I) \in F$ such that $u_i > w_i$ for all i. Since $v_i \geqslant w_i$, obviously, any regular game is weakly regular.[9]

While the following result is a corollary to a more general result below, we begin with it for the sake of clarity.

Theorem 3. *For any weakly regular game,*

$$\lim_{\varepsilon\downarrow 0,\delta\uparrow 1}{}^{\infty}\ \mathcal{U}(\varepsilon,\delta) = F^* \cap W.$$

That is, the limiting payoff set is the set of feasible payoffs (taking into account the ability to dissipate payoffs by switching actions) which are weakly individually rational.

Thus we have a simple characterization of equilibrium payoffs in the two extreme cases, where $\varepsilon/(1-\delta)$ converges to 0 and where it converges to ∞. As one might expect, the middle ground is more complex. Our next result shows that the transition between these extremes is gradual in the sense that as k increases, we gradually fill in all the payoffs in $F^* \cap W$ which are not in $F \cap R$. More precisely,

Theorem 4. *For any weakly regular game and any $\eta > 0$, there is a k_η such that for all $k \geqslant k_\eta$, for all $u \in F^* \cap W$, there is a u' within η of u with*

$$u' \in \lim_{\varepsilon\downarrow 0,\delta\uparrow 1}{}^{k}\ \mathcal{U}(\varepsilon,\delta).$$

As $\eta \downarrow 0$, $k_\eta \to \infty$.

Thus the set of limiting equilibrium payoffs when $\varepsilon/(1-\delta)$ converges to k approximates $F^* \cap W$ with the precision of the approximation improving as we increase k. However, to generate the entire set, we need $k \to \infty$ in general.

A natural question to ask is how large the limiting set of equilibria is for "moderate" values of k. If the set differs from the usual Folk Theorem set only when k is extremely large, the interest in Theorems 3 and 4 is somewhat limited. The next result shows that we only need the switching costs to be on the order of two periods worth of payoff to generate a significant difference from the usual Folk Theorem. More specifically, we have

Theorem 5. *For any weakly regular game and any*

$$k > 2\max_i\Big[\max_{a\in A} u_i(a) - w_i\Big],$$

we have

$$F \cap W \subseteq \lim_{\varepsilon\downarrow 0,\delta\uparrow 1}{}^{k}\ \mathcal{U}(\varepsilon,\delta).$$

That is, all feasible (in the traditional sense) and weakly individually rational payoffs are included in the limiting equilibrium set for "moderate" k.

To understand our interpretation of the condition on k, suppose we have a sequence $(\varepsilon_n, \delta_n)$ converging to $(0, 1)$ with $\varepsilon_n/(1-\delta_n)$ converging to a k satisfying the condition of Theorem 5. For large n, $\varepsilon_n/(1-\delta_n)$ is very close to k, so

$$\varepsilon_n > (1-\delta_n)2\max_i\Big[\max_{a\in A} u_i(a) - w_i\Big].$$

[9] Note that it would be equivalent to define weak regularity using F^* in place of F.

The left-hand side is the cost of a change of actions in the current period. The right-hand side is approximately the gain in payoff over two periods from moving from w_i to $\max_{a\in A} u_i(a)$ for some player i. As we will see shortly, w_i is the lowest payoff that can be imposed on a player, so this payoff gain is, roughly speaking, the large "plausible" payoff gain to a change in actions. In this sense, this condition says that the switching cost is bigger than the largest potentially relevant two-period payoff gain.

The proofs of these results are in the Appendix, but here we sketch the idea. It is obvious that for any k, the limiting payoff set is contained in $F^* \cap W$, so the critical issue is when these payoffs can be generated by some equilibrium.

First, we need to establish that W is the appropriate version of individual rationality. To see this, consider the payoff a player receives if the others are trying to minimize his payoff. If the other players continually move to the action which minimizes his payoff given the action he has most recently played, he will either stop changing actions and get his pure reservation payoff or change actions every period. If $\varepsilon/(1-\delta)$ converges to a large enough number, these switching costs become too large for this second option to be optimal. As it turns out, the bound on k given in Theorem 5 is sufficient to establish this. Hence when this condition holds, we can force a player down to his pure reservation payoff.

To see that F^* is the appropriate definition of feasibility is a little more complex. Suppose we wish to construct strategies generating a particular payoff vector u in F^*. Any such payoff vector can be written in the form

$$u = \sum_{a\in A} \alpha(a)u(a) - x$$

where α is a probability distribution over A and x is a vector of costs. It is tedious but not difficult to show that for large enough k, any such payoff can be approximately generated by constructing an appropriate cycle of actions. The cycle is chosen so that the relative frequencies of actions over the cycle approximates α and the relative frequency of changes of actions over the cycle generates x. To illustrate the latter, suppose, for example, that the cycle is of length N and that player i changes actions in the first N_i periods of the cycle only. Then his switching costs over the entire infinite horizon are

$$\varepsilon\left[\sum_{t=0}^{N_i-1}\delta^t\right]\left[\sum_{t=0}^{\infty}\delta^{Nt}\right] = \frac{\varepsilon}{1-\delta}\left[\sum_{t=0}^{N_i-1}\delta^t\right]\frac{1-\delta}{1-\delta^N} = \frac{\varepsilon}{1-\delta}\left[\frac{\sum_{t=0}^{N_i-1}\delta^t}{\sum_{t=0}^{N-1}\delta^t}\right].$$

The second term in the last expression converges to N_i/N as $\delta \to 1$. Hence if we take the limit as $(\varepsilon,\delta) \to (0,1)$ along a sequence for which $\varepsilon/(1-\delta) \to k$, we see that the switching costs converge to $k(N_i/N)$. Hence by setting the frequency of i's action changes over the cycle appropriately, we can generate whatever switching cost is needed. Of course, if k is too small, this may require $N_i/N > 1$ which is impossible. Even if k is large enough that $N_i/N < 1$, if k is small, setting the appropriate N_i/N can conflict with setting the frequencies of actions to match α. However, for large enough k, we avoid this problem. In short, this cycle can be chosen so that as $(\varepsilon,\delta) \to (0,1)$, the payoff converges to approximately u, where the approximation can be made arbitrarily close for large enough k. Thus any payoff in F^* is feasible even in the limit as $(\varepsilon,\delta) \to (0,1)$.

Given these two facts, the completion of the proofs of Theorems 3 and 4 is similar to but more complex than the Folk Theorem construction of Fudenberg and Maskin (1986). To explain why it is more complex, we briefly sketch the argument. In Fudenberg and Maskin, if we wish to find an equilibrium generating payoff vector u, there must be some joint randomization (using the public randomizing device) which yields this payoff in every period. To ensure that agents play this every period, we set up a system of punishments. Specifically, if player i deviates, he is minmaxed for a certain number of periods and then agents play a joint randomization which yields exactly the payoff u except that i receives a small amount ν less. If any agent deviates from the punishment or the post–punishment play, this agent is then punished in the same fashion. These reductions by ν ensure that no player wishes to deviate from the equilibrium, that no player $j \neq i$ wishes to deviate from the punishment of i, and that no player $j \neq i$ wishes to deviate after i's punishment. Since i is punished by being minmaxed, he has no profitable deviation while being punished. Finally, the length of time i is punished is chosen to ensure that i does not deviate after being punished.

Our equilibrium works the same way but with several complications. First, we cannot choose a joint randomization to achieve an exact payoff because of the way the switching costs complicate such payoff calculations, as discussed above. Hence we have an "approximation error" term to keep track of, not just u_i's and ν's. This complicates a few expressions, but creates no conceptual difficulties.

Second, we typically must generate payoffs via a cycle rather than a fixed joint randomization. Hence deviations can occur at any point in the cycle. Since, for example, it is possible that a player earns a significantly worse payoff in the second half of a cycle than the first, he may have a different incentive to deviate midway through a cycle than at the beginning. Again, this complicates various payoff calculations but leads to no conceptual issues. In particular, this factor simply forces us to compute the length of punishment differently than in Fudenberg and Maskin.

Third, we must consider the possibility that a deviation leads to a lower game payoff but to fewer changes of actions, giving a higher payoff on balance. Of course, this consideration is not present in the usual Folk Theorems. It occasionally requires us to be more careful than usual in checking of deviations.

Finally, in Fudenberg and Maskin, a player is minmaxed for punishment. Hence, even if he optimizes against the punishment, his payoff is the reservation utility. But with the pure reservation utility we must use, a player who is being punished is not choosing a best reply to the punishment actions. Hence we must ensure that a player cooperates with his own punishment. As discussed above, the switching costs enable us to force a player down to his pure reservation utility and hence play a role in our proof which has no obvious parallel in Fudenberg and Maskin's proof.

To explain Theorem 5, recall our earlier comment that if k is too small, some payoffs in $F^* \cap W$ are not feasible. However, this problem arises only when we want to construct payoffs in F^* which are not in F. If a payoff is in F, we do not need the switching to dissipate payoffs. In fact, for *any* k, we can find a cycle which approximately generates any payoff $u \in F$ arbitrarily well. This fact together with the observation above that the condition in Theorem 5 is sufficient to make w the relevant notion of individual rationality explains why Theorem 5 holds.

We obtain a more unusual characterization in the case of the limit of $\mathcal{U}_\infty(\varepsilon)$ as $\varepsilon \downarrow 0$. Let $U_\geqslant$ denote those points in U which are greater than w. That is,

$$U_\geqslant = \{u = (u_1, \ldots, u_I) \in U \mid u_i \geqslant w_i, \ \forall i\}.$$

For the next result, we need an assumption which we call *rewardability*. We say that a game satisfies rewardability if for each i, there is a payoff vector $u^i = (u^i_1, \ldots, u^i_I) \in U_\geqslant$ with $u^i_i > w_i$. It is worth emphasizing that this property is much stronger than regularity.

To see the idea behind the name, suppose this assumption does not hold. As mentioned in the introduction, in $G_\infty(\varepsilon)$, the only u vectors which can be achieved infinitely often with positive probability are those in $U_\geqslant$. If for some player i, all these vectors give him w_i, then he cannot be rewarded for aiding in the punishment of a deviator. This complication restricts the set of equilibria in a complex fashion as we explain in more detail below.

We also require a weak form of a genericity assumption. Specifically, we say that a game is *generic* if for all i, there is a unique action $\bar{a}_i$ such that $u_i(\bar{a}_i, a_{\sim i}) = w_i$ for some $a_{\sim i} \in A_{\sim i}$. That is, each player has a unique maxmin action. Obviously, this is implied by the more customary form of a genericity assumption which would say that $a \neq a'$ implies $u_i(a) \neq u_i(a')$ for all i.

Theorem 6. *For a generic game satisfying rewardability,*

$$\lim_{\varepsilon \downarrow 0} \mathcal{U}_\infty(\varepsilon) = c(U_\geqslant) \cap W.$$

To see how this differs from the payoff set from Theorem 3, note that we can write that set as $c(U) \cap W$. In this form, the difference is obvious: the payoff set of Theorem 6 only puts weight on payoffs which are weakly individually rational, not all those generated by pure strategies.

The proof that any payoff in $c(U_\geqslant) \cap W$ is a limiting equilibrium payoff is similar to standard Folk Theorem arguments. The more unusual part of the proof is the demonstration that no payoff outside this set can be close to an equilibrium payoff. We sketch the idea in the context of the Prisoners' Dilemma we used in the introduction:

	C	D
C	3, 3	0, 4
D	4, 0	2, 2

Let $\mathcal{P}(C, C)$ denote the set of infinite sequences of actions which eventually "absorb" at (C, C)—that is, sequences with the property that for some T, the actions played at any $t \geqslant T$ are (C, C). Define $\mathcal{P}(C, D)$, etc., analogously. Note that any sequence of actions which is not in $\mathcal{P}(C, C)$, $\mathcal{P}(C, D)$, $\mathcal{P}(D, C)$, or $\mathcal{P}(D, D)$ has at least one player changing actions infinitely often. If any player has a positive probability of switching actions infinitely often, his expected payoff is $-\infty$ and so his strategy cannot be optimal. Hence any equilibrium has to put zero probability on such an event. That is, the sets $\mathcal{P}(C, C)$, $\mathcal{P}(C, D)$, $\mathcal{P}(D, C)$, and $\mathcal{P}(D, D)$ must have probability 1 in total. The main claim of Theorem 6 is that the sets $\mathcal{P}(C, D)$ and $\mathcal{P}(D, C)$ must have zero probability in equilibrium.

To see this, suppose, say, $\mathcal{P}(C, D)$ has probability $\mu > 0$. Clearly, it cannot have probability 1. If it did, player 1's payoff in equilibrium would be 0, while playing a constant action of D gives him a payoff of 2, a contradiction. Clearly, too, there can be no history with the property that the probability of $\mathcal{P}(C, D)$ conditional on this history is 1. If it were, then for any switching cost less than 2, player 1 could profitably deviate on that history to a constant action of D and be better off.

What is not so transparent is whether it is possible to construct the public randomizations in such a way that play does absorb at (C, D) but this fact is hidden from player 1. In other words, can we construct strategies with the property that there is a positive probability that (C, D) is played from a certain point onward and yet along this path of play, player 1 always believes there is a nontrivial probability that some other action will be played in the future?

In fact, the answer to this question is no. To see this, suppose (C, D) is played at period t and consider the probability player 1 gives to the event $\mathcal{P}(C, D)$ conditional on this fact. Clearly, any path of play which absorbs at a different action profile at a period before t must have zero probability at this point. Hence for large t, the conditional probability that the play path is $\mathcal{P}(C, C)$, $\mathcal{P}(D, C)$, or $\mathcal{P}(D, D)$ must be getting small. At the same time, this fact that (C, D) is played at t cannot rule out the possibility that play has already absorbed at (C, D). Hence as t gets large, the conditional probability on $\mathcal{P}(C, D)$ must converge to 1. But once this conditional probability is large enough, player 1 will certainly deviate to D, a contradiction. Note that this argument actually implies that we cannot have a *Nash* equilibrium putting positive probability on $\mathcal{P}(C, D)$, much less a subgame perfect equilibrium.

Remark 3. To see what happens with games which violate rewardability, consider the following game:

	L	R
U	$0, 1$	$1, -2$
D	$-2, 0$	$-1, -1$

It is not hard to see that $w_1 = w_2 = 0$ so $U_{\geqslant} = \{(0, 1)\}$. Hence rewardability fails because the only vector in $U_{\geqslant}$ gives player 1 his pure reservation payoff. For this game,

$$c(U_{\geqslant}) \cap W = \{(0, x) \mid 0 \leqslant x \leqslant 1\}.$$

However, the unique equilibrium payoff is $(0, 1)$. Intuitively, this is because player 1 must get a payoff of 0 in any subgame perfect equilibrium. Hence he cannot be induced to change actions and so will not punish 2 for deviations. Hence 2 must receive a payoff of 1. It is not hard to see how one could give a characterization of the limiting equilibrium set without rewardability. Analogously to Wen (1994), one can explicitly work out the way in which punishment is constrained to give an exact characterization of the limiting equilibrium set. More specifically, if for some player i, every vector in $U_{\geqslant}$ gives him a payoff of w_i, then he must play a fixed action at every history of every equilibrium. We can set this player to the constant action he must play and solve the "reduced game" among the remaining players, iterating this procedure as necessary.

Remark 4. It is worth noting that the proof of Theorem 6 also shows that the reduction in the set of payoffs is not entirely a "vanishing ε" phenomenon. More specifically, the proof shows that there is a $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, $\mathcal{U}_\infty(\varepsilon)$ is contained in $c(U_{\geqslant}) \cap W$.

It is natural to wonder why we get such a dramatic difference between the discounting and limit of means cases. This is much more than the dimensionality issue that comes up in the analysis of repeated games without switching costs. The difference here hinges, as with most of our results, on the relationship between the switching cost and the length of a period. To see the point, consider the discounting case and suppose we take the limit of $\mathcal{U}(\varepsilon, \delta)$ as $(\varepsilon, \delta) \to (0, 1)$ along a sequence where $\varepsilon/(1 - \delta) \to \infty$. As argued above, we can think of this as making the period length short relative to the switching cost. However, this effect can be undone in equilibrium. To see the point, note that we could always construct equilibria in which the players act as if a block of k periods was only one period. That is, they only change actions at intervals of k periods.[10] By constructing such equilibria, we can effectively make the length of a period arbitrarily long relative to the switching cost.

For any $\delta < 1$, this matters. However, in the limit of means case, it does not. In $G_\infty(\varepsilon)$, only the number of times the players change actions matters, not the intervals at which these changes occur. Hence this is the only situation where the players cannot endogenously alter the relationship between switching costs and payoffs in "a period."[11]

4. Continuity

To say whether our results indicate a discontinuity in the equilibrium outcome correspondence, we must first define a notion of convergence of games. We say that a sequence of games G^n converges to a game G if for every player and every sequence of action profiles, the payoff in G^n converges as $n \to \infty$ to the payoff in G. That is, we say that

$$\lim_{n\to\infty} G(\varepsilon_n, \delta_n) = G(\varepsilon, \delta)$$

iff for all i and all $(a^0, a^1, \ldots) \in A^\infty$

$$\lim_{n\to\infty}\left[(1-\delta_n)\sum_{t=0}^{\infty}\delta_n^t u_i\left(a^t\right) - \sum_{t=0}^{\infty}\delta_n^t \varepsilon_n I_i\left(a^{t-1}, a^t\right)\right] = (1-\delta)\sum_{t=0}^{\infty}\delta^t u_i\left(a^t\right) - \sum_{t=0}^{\infty}\delta^t \varepsilon I_i\left(a^{t-1}, a^t\right).$$

We define convergence to $G_\infty(\varepsilon)$ or convergence of the sequence $G_\infty(\varepsilon_n)$ analogously. We emphasize that we allow convergence of a payoff to $-\infty$ in this definition. That is, we define the limit of a monotonically decreasing sequence with no lower bound as $-\infty$. In particular, any sequence of actions where some player changes actions every period has a payoff of $-\infty$ for that player in $G_\infty(\varepsilon)$. Hence if some sequence of games is to converge to $G_\infty(\varepsilon)$, we must allow a sequence of payoffs to converge to $-\infty$.[12]

Lemma 1. *For any sequence $(\varepsilon_n, \delta_n)$, we have*

$$\lim_{n\to\infty} G(\varepsilon_n, \delta_n) = \lim_{n\to\infty} G(0, \delta_n)$$

if and only if $\varepsilon_n/(1-\delta_n) \to 0$.

Proof. Fix any i and any sequence of actions $a^0, a^1, \ldots$ where i changes actions every period. Note that the payoff to i from this sequence in $G(0, \delta_n)$ is bounded between $\min_{a\in A} u_i(a)$ and $\max_{a\in A} u_i(a)$. Hence as $n \to \infty$, the payoff cannot converge to $-\infty$. Hence if the payoff in $G(0, \delta_n)$ has the same limit as $n \to \infty$ as the payoff in $G(\varepsilon_n, \delta_n)$, the latter must also be finite, so the difference in payoffs must converge to 0 as $n \to \infty$. Note that the payoff to i in $G(0, \delta_n)$ minus the payoff in $G(\varepsilon_n, \delta_n)$ is

$$\sum_{t=0}^{\infty}\delta_n^t \varepsilon_n = \frac{\varepsilon_n}{1-\delta_n}.$$

Hence if $\varepsilon_n/(1-\delta_n) \not\to 0$, $G(\varepsilon_n, \delta_n)$ and $G(0, \delta_n)$ cannot have the same limit.

[10] Of course, one cannot prevent players from deviating from this and changing actions more frequently. However, the punishment for deviations can come more quickly as well.

[11] To be more precise, in *our* limit of means formulation, players cannot effectively alter the length of a period. As discussed in the introduction, Chakrabarti (1990) integrates switching costs into a limit of means set up differently than we do. In his analysis, treating blocks of periods as a single period and thus effectively lengthening a period is feasible and plays a key role in proofs. Note that he obtains the same results we obtain with our discounting formulation, supporting the view that it is the inability to endogenously alter period length that gives the difference between our limit of means and discounting cases.

[12] Because of this, our definition of convergence of a sequence of games is not the same as that generated by defining the distance between two games to be the supremum payoff difference over players and sequences of action profiles. In particular, Lemma 3 below would not hold under this alternative definition. To see this, note that every payoff in $G_f(\varepsilon, \Delta)$ is finite. Fix any sequence of actions where player i changes actions infinitely often. Then the difference in the payoffs to i from this sequence between $G_f(\varepsilon, \Delta)$ and $G_\infty(\varepsilon)$ is ∞ for every $\varepsilon > 0$ and $\Delta > 0$. Hence as $\Delta \to 0$, the distance between $G_f(\varepsilon, \Delta)$ and $G_\infty(\varepsilon)$ according to this definition does not go to zero.

For the converse, fix any i and any sequence of actions $a^0, a^1, \ldots,$ not necessarily one where i changes actions every period. Then the payoff in $G(0, \delta_n)$ minus the payoff in $G(\varepsilon_n, \delta_n)$ is

$$\sum_{t=0}^{\infty} \delta_n^t \varepsilon_n I_i\left(a^t, a^{t-1}\right) \leqslant \frac{\varepsilon_n}{1-\delta_n}.$$

Hence if $\varepsilon_n/(1-\delta_n) \to 0$, $G(\varepsilon_n, \delta_n)$ and $G(0, \delta_n)$ do have the same limit. □

Given this result, we see that Theorems 3, 4, and 5 do not indicate a discontinuity at $(\varepsilon, \delta) = (0, 1)$ in the equilibrium outcome correspondence. In particular, in the class of games considered, the limiting set of payoffs in $G(\varepsilon_n, \delta_n)$ and $G(0, \delta_n)$ as $n \to \infty$ differ only if the limiting games differ.

That Theorem 6 does not imply a discontinuity at $\varepsilon = 0$ is a corollary to

Lemma 2. *Fix any strictly positive sequence ε_n. Then*

$$\lim_{n\to\infty} G_\infty(\varepsilon_n) \neq G_\infty(0).$$

Proof. Fix any i and any sequence of actions where i changes actions infinitely often. i's payoff in $G_\infty(\varepsilon_n)$ is $-\infty$ for every n, while his payoff in $G_\infty(0)$ is bounded from below by $\min_{a\in A} u_i(a)$. Hence i's payoff in $G_\infty(\varepsilon_n)$ does not converge to his payoff in $G_\infty(0)$ as $n \to \infty$. □

Remark 5. It is common to describe the infinitely repeated game with the limit of means criterion as the limit of the game with discounting as $\delta \to 1$. Our definition of convergence does not support this view. The reason is simply that, as is well known, there are sequences of action profiles for which the limiting average payoff does not exist and the liminf of the average payoff is not equal the limiting discounting payoff as $\delta \to 1$. On the other hand, if we define the limit of means game to be the limit as $\delta \to 1$ of $G(\varepsilon, \delta)$, then our results imply a discontinuity in δ for a fixed $\varepsilon > 0$. Specifically, for $\hat{\varepsilon} > 0$ small enough, we have

$$\lim_{\delta\uparrow 1} \mathcal{U}(\hat{\varepsilon}, \delta) \neq \mathcal{U}_\infty(\hat{\varepsilon}).$$

This holds simply because for $\hat{\varepsilon}$ sufficiently small, the left-hand side is close to

$$\lim_{\varepsilon\downarrow 0, \delta\uparrow 1}{}^{\infty}\ \mathcal{U}(\varepsilon, \delta) = F^* \cap W,$$

while the right-hand side is close to

$$\lim_{\varepsilon\downarrow 0} \mathcal{U}_\infty(\varepsilon) = c(U_\geqslant) \cap W,$$

and these sets are far apart in general. So if we define the limit as $\delta \to 1$ of the discounted game as the limit of means game, this result says that there is a discontinuity at $\delta = 1$. It is well known that there are stage games for which $\lim_{\delta\uparrow 1} \mathcal{U}(0, \delta) \neq \mathcal{U}_\infty(0)$. However, the discontinuity for $\varepsilon > 0$ differs from the known discontinuity for $\varepsilon = 0$ in two ways. First, the known discontinuity is ruled out by fairly weak conditions such as the dimensionality condition we used in Theorem 1.B. The discontinuity for $\varepsilon > 0$ is not ruled out by such conditions. Second, the known discontinuity is a failure of lower semicontinuity, not upper. That is, the usual discontinuity occurs when $\mathcal{U}_\infty(0)$ is strictly larger than $\lim_{\delta\uparrow 1} \mathcal{U}(0, 1)$. Under very weak conditions, $\mathcal{U}_\infty(0)$ equals $F \cap R$ and it is *always* true that $\mathcal{U}(0, \delta) \subseteq F \cap R$. Hence we typically get upper semicontinuity in δ at $(\varepsilon, \delta) = (0, 1)$. By contrast, when the discontinuity in δ for fixed $\varepsilon > 0$ occurs, it is because $\mathcal{U}_\infty(\varepsilon)$ is close to a set which is strictly *smaller* than a set close to $\lim_{\delta\uparrow 1} \mathcal{U}(\varepsilon, \delta)$. Hence we generally violate upper semicontinuity.

Finally, we can use the results here to determine whether the results in Lipman–Wang (2000) indicate a discontinuity in the equilibrium outcome correspondence. To do so, we embed our previous model into this class of games in order to define convergence of that model. To be more precise, let $G_f(\varepsilon, \Delta)$ denote the infinitely repeated stage game where player i evaluates a sequence of actions $a^0, a^1, \ldots$ by the criterion

$$\sum_{t=0}^{T(\Delta)} \left[\Delta u_i\left(a^t\right) - \varepsilon \#\left\{t \leqslant T(\Delta) \mid a_i^t \neq a_i^{t-1}\right\}\right]$$

where $T(\Delta)$ is the largest integer satisfying $(T+1)\Delta \leqslant \mathcal{L}$. (Recall that the length of time the game is played is equal to $\mathcal{L}$.) In other words, while the game is infinitely repeated, only actions played in the first $T(\Delta)+1$ periods matter.[13] Given this, we can define convergence of a sequence of such games just as before.[14] In particular, it is easy to show that

Lemma 3. *For any $\varepsilon > 0$ and any sequence Δ_n with $\Delta_n > 0$ and converging to 0,*

$$\lim_{n\to\infty} G_f(\varepsilon, \Delta_n) = G_\infty(\varepsilon).$$

On the other hand, analogously to Lemma 1, we have

Lemma 4. *Given a strictly positive sequence $(\varepsilon_n, \Delta_n)$,*

$$\lim_{n\to\infty} G_f(\varepsilon_n, \Delta_n) = \lim_{n\to\infty} G_f(0, \Delta_n)$$

if and only if $\varepsilon_n/\Delta_n \to 0$.

Proof. Fix any i and any sequence of action profiles where i changes actions every period. Since i's payoff in $G_f(0, \Delta_n)$ is bounded from below by $\min_{a\in A} u_i(a)$, the limit of his payoff as $n \to \infty$ must be finite. Hence if the payoff in $G_f(0, \Delta_n)$ has the same limit as $n \to \infty$ as the payoff in $G_f(\varepsilon_n, \Delta_n)$, the latter must also be finite, so the difference in payoffs must converge to 0 as $n \to \infty$. Note that the payoff to i in $G_f(0, \Delta_n)$ minus the payoff in $G_f(\varepsilon_n, \Delta_n)$ is $\varepsilon_n T(\Delta_n)$. But

$$\lim_{n\to\infty} \varepsilon_n T(\Delta_n) = \lim_{n\to\infty} \frac{\varepsilon_n}{\Delta_n} \Delta_n T(\Delta_n).$$

By definition of $T(\Delta_n)$ and the fact that $\Delta_n \to 0$, we must have $\lim_{n\to\infty} \Delta_n T(\Delta_n) = \mathcal{L}$. Hence $\lim_{n\to\infty} \varepsilon_n T(\Delta_n) = 0$ iff $\lim_{n\to\infty} \varepsilon_n/\Delta_n = 0$. So if $G_f(\varepsilon_n, \Delta_n)$ and $G_f(0, \Delta_n)$ have the same limit, it must be true that $\varepsilon_n/\Delta_n \to 0$.

For the converse, fix any i and any sequence of actions $a^0, a^1, \ldots,$ not necessarily one where i changes actions every period. Then the payoff in $G(0, \Delta_n)$ minus the payoff in $G(\varepsilon_n, \Delta_n)$ is

$$\varepsilon_n \#\{t \leqslant T(\Delta) \mid a_i^t \neq a_i^{t-1}\} \leqslant \varepsilon_n T(\Delta_n).$$

If $\varepsilon_n/\Delta_n \to 0$, then $\varepsilon_n T(\Delta_n) \to 0$, so the payoff difference goes to zero. Hence if $\varepsilon/\Delta_n \to 0$, $G_f(\varepsilon_n, \Delta_n)$ and $G_f(0, \Delta_n)$ do have the same limit. □

Just as with $G(\varepsilon, \delta)$, this result implies that there is no discontinuity in ε. As discussed in Lipman–Wang (2000), $G_f(\varepsilon, \Delta)$ yields different results from the usual finitely repeated game only when $\Delta < K\varepsilon$ for some $K > 0$. In particular, if ε/Δ is small, the switching costs do not change the usual results. In other words, when $\mathcal{U}_f(\varepsilon, \Delta)$ for small ε is significantly different from $\mathcal{U}_f(0, \Delta)$, it must be true that Δ is small relative to ε and so $G_f(\varepsilon, \Delta)$ is far from $G_f(0, \Delta)$. Hence there is no discontinuity in ε at $\varepsilon = 0$.

On the other hand, for any strictly positive ε, in general, $\mathcal{U}_f(\varepsilon, \Delta)$ is discontinuous in Δ at $\Delta = 0$. The easiest way to see the point is to return to the Prisoners' Dilemma game discussed earlier:

	C	D
C	3, 3	0, 4
D	4, 0	2, 2

As we noted, for any small $\varepsilon > 0$, it is impossible to sustain an equilibrium in $G_\infty(\varepsilon)$ which puts positive weight on (0, 4) or (4, 0). Hence the set of equilibrium payoffs is a subset of the (u_1, u_2) such that $2 \leqslant u_i \leqslant 3$ for both i. On the

[13] This method of embedding a finitely repeated game into the infinitely repeated one is similar to that used by Fudenberg and Levine (1983). They used a fixed action after some period, an approach less convenient for our purposes.

[14] To be more precise, the definition changes slightly in that we must replace lim with lim inf.

other hand, it is easy to use Theorem 1 of Lipman–Wang (2000) to construct equilibrium payoffs outside this set for $G_f(\varepsilon, \Delta)$ as $\Delta \downarrow 0$.[15]

This implies that for ε sufficiently small,

$$\lim_{\Delta\downarrow 0} \mathcal{U}_f(\varepsilon, \Delta) \neq \mathcal{U}_\infty(\varepsilon)$$

even though

$$\lim_{\Delta\downarrow 0} G_f(\varepsilon, \Delta) = G_\infty(\varepsilon)$$

by Lemma 3. Hence we have a discontinuity in Δ at $\Delta = 0$ for any sufficiently small $\varepsilon > 0$.

For the case of $\varepsilon = 0$, there is already a well-known discontinuity as $\Delta \downarrow 0$. Recall that $G_f(0, \Delta)$ is just the usual finitely repeated game where $T(\Delta)$ is the number of repetitions and $G_\infty(0)$ is the usual infinitely repeated game with the limit of means payoff criterion. So the well-known difference between finitely repeated and infinitely repeated games corresponds to a discontinuity in the equilibrium payoff correspondence at $\Delta = 0$.

There are two ways to see that the discontinuity in Δ for $\varepsilon > 0$ is fundamentally different from the known discontinuity for $\varepsilon = 0$. First, the known discontinuity is a failure of lower semicontinuity, not upper. That is, in the known discontinuity, the set of equilibrium payoffs at the limit is larger than the limiting set. Above, we showed that the limiting set of equilibrium payoffs contains points not in the set at the limit. Hence the discontinuity when $\varepsilon > 0$ shows a failure of upper semicontinuity, not lower.

Second, the discontinuity for $\varepsilon > 0$ occurs in some games where there is no discontinuity for $\varepsilon = 0$. For example, consider the coordination game:

	L	R
U	4, 4	1, 0
D	0, 1	2, 2

This game has no discontinuity in Δ at $\varepsilon = 0$ since it satisfies the Benoit–Krishna (1985) conditions for a finite repetition Folk Theorem. On the other hand, Theorem 4 of Lipman–Wang implies that for all sufficiently small ε,

$$\lim_{\Delta\downarrow 0} \mathcal{U}_f(\varepsilon, \Delta) = \{(4, 4)\},$$

while Theorem 6 above shows that $\mathcal{U}_\infty(\varepsilon)$ is close to the set of all (u_1, u_2) with $1 \leqslant u_i \leqslant 4$, $i = 1, 2$. Hence there is a discontinuity in Δ for ε small but positive. Note that this is a failure of lower semicontinuity, so the equilibrium outcome correspondence $\mathcal{U}_f(\varepsilon, \Delta)$ is neither upper nor lower semicontinuous in Δ at $\Delta = 0$ and $\varepsilon > 0$.

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Appendix A. Proof of Theorem 4

It is obvious that no $u \notin F^* \cap W$ can be an equilibrium payoff. Such a u is either infeasible or has some player with a lower payoff than what he could guarantee himself by a constant action. Hence we only need to show that all payoffs in $F^* \cap W$ are close to equilibrium payoffs for k sufficiently large.

We begin by noting that every such payoff can be approximately generated by a cycle of actions.

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[15] More specifically, Theorem 1 of Lipman–Wang implies that for this game, there is a $K > 0$ such that for all sufficiently small ε and all $\Delta \in (0, K\varepsilon)$, there is a subgame perfect equilibrium in $G_f(\varepsilon, \Delta)$ where both players cooperate in every period. Given this, we can construct an equilibrium which begins with (C, D) played for some number of periods, followed by (C, C) for the rest of the game with play moving to (D, D) in the event of deviation. As long as the fraction of the time spent at (C, D) is small enough that player 1 gets a payoff of at least 2, this will be a subgame perfect equilibrium. In particular, then, the fraction of time spent at (C, D) does not have to converge to 0 as $\Delta \downarrow 0$.

Lemma 5. *Fix any $\eta > 0$ and any $u = (u_1, \ldots, u_I) \in F^* \cap W$. Then there exists $\bar{k}_\eta$, independent of u, such that for all $k \geqslant \bar{k}_\eta$, there is a finite cycle of action profiles such that for any sequence $(\varepsilon_n, \delta_n) \to (0, 1)$ with $\varepsilon_n/(1-\delta_n) \to k$, the payoff to i along this sequence converges as $n \to \infty$ to within η of u_i. As $\eta \downarrow 0$, $\bar{k}_\eta \to \infty$.*

While this lemma is unsurprising, the proof is quite lengthy and so is omitted. See the working paper, Lipman and Wang (2006), for a complete proof.

By weak regularity, any $u \in F^* \cap W$ is arbitrarily close to some $\bar{u} \in F^* \cap W$ with $\bar{u}_i > w_i$ for all i. Hence we may as well restrict attention to payoffs which are strictly above w.

So fix any $u \in F^* \cap W$ with $u_i > w_i$ for all i. Fix any $\eta > 0$ such that $3\eta < u_i - w_i$ for all i. Let k be any number larger than the $\bar{k}_\eta$ of Lemma 5 and

$$2 \max_i \Big[\max_{a \in A} u_i(a) - w_i \Big].$$

Since $\bar{k}_\eta \to \infty$ as $\eta \downarrow 0$, obviously, k must also converge to infinity as $\eta \downarrow 0$.

Fix any sequence $(\varepsilon_n, \delta_n)$ converging to $(0, 1)$ with $\varepsilon_n/(1-\delta_n) \to k$. By Lemma 5, there is a finite cycle of actions, independent of n, which generates a payoff vector, say $u(n)$, such that

$$u_i - \eta \leqslant \lim_{n\to\infty} u_i(n) \leqslant u_i + \eta, \quad \forall i.$$

We now show that for every sufficiently large n, there is a subgame perfect equilibrium with an equilibrium path equal to this cycle of actions. The subgame perfect equilibrium strategies are independent of n.

Fix any ν such that $2\eta < \nu < \min_i(u_i - w_i) - \eta$. (Since $3\eta < \min_i(u_i - w_i)$, such ν's exist.) By the definition of F^*, for all i, $u^i = (u_1, \ldots, u_{i-1}, u_i - \nu, u_{i+1}, \ldots, u_I) \in F^* \cap W$. From Lemma 5, for each i, there is a finite cycle such that the payoffs along the sequence converges as $n \to \infty$ to within η of u^i. Without loss of generality, we choose these cycles so that the cycle generating the payoff near u^i does not begin with i playing one of his maxmin actions. Let $u^i(n) = (u^i_1(n), \ldots, u^i_I(n))$ denote the payoff along this cycle given $(\varepsilon_n, \delta_n)$, so

$$u_j - \eta \leqslant \lim_{n\to\infty} u^i_j(n) \leqslant u_j + \eta, \quad \forall j \neq i$$

and

$$u_i - \nu - \eta \leqslant \lim_{n\to\infty} u^i_i(n) \leqslant u_i - \nu + \eta.$$

The strategies we use are essentially the same as those used by Fudenberg and Maskin (1986) in their proof of their Theorem 2. Play begins in phase I. In phase I, the players follow the cycle of actions generating the payoff near u. As long as no player deviates, all follow these actions. If there is a unilateral deviation by player i in phase I, we move to phase II_i. In this phase, i plays any one of his maxmin actions for M_i periods (M_i characterized below). In each period of phase II_i, all players other than i play some vector of actions which minimizes i's payoff against his action in the preceding period. (Given that i follows his equilibrium strategy in this phase, these actions will be those which minimize i's payoff from his chosen maxmin action from the second period of phase II_i onward.) At the end of these M_i periods, we move to phase III_i. In this phase, the players follow the cycle of actions which generates payoffs near u^i. If any player j unilaterally deviates in phase II_i or III_i, we move to phase II_j and then III_j. (In particular, if player i unilaterally deviates in phase II_i, we restart the phase.) We assign an arbitrary subgame perfect continuation if there are multiple simultaneous deviations in any phase.

M_i is set to any integer M such that

$$u_i - \nu - \eta > \frac{L_i + 1}{M + L_i} \max_{a \in A} u_i(a) + \frac{M - 1}{M + L_i}\, w_i$$

where L_i is the length of the cycle of actions in phase III_i. By construction, $\nu < u_i - w_i - \eta$, so $u_i - \nu - \eta > w_i$, implying that such an M_i exists.

To see that these strategies form a subgame perfect equilibrium, consider any player i in phase I. If i deviates and then follows the equilibrium strategies, his expected payoff must be no larger than

$$(1 - \delta_n) \sum_{t=0}^{M_i} \delta_n^t \max_{a \in A} u_i(a) + \delta_n^{M_i+1} u^i_i(n).$$

As $n \to \infty$, this converges to no more than $u_i - \nu + \eta$. By contrast, if there were ℓ periods left in the cycle and i did not deviate, his payoff would be at least

$$\sum_{t=0}^{\ell-1} \delta_n^t \Big[(1-\delta_n)\min_{a\in A} u_i(a) - \varepsilon_n\Big] + \delta_n^\ell u_i(n)$$

which converges to at least $u_i - \eta$ as $n \to \infty$. Hence $\nu > 2\eta$ implies that, for n sufficiently large, no player can gain by deviating in phase I.

A similar argument applies to any player $j \neq i$ and a deviation in phase II_i or III_i. By deviating, a player may gain over a finite number of periods, but has a loss of at least approximately $\nu - 2\eta$ over the infinite horizon after that. For n large enough, the loss must exceed the gain.

The case of a deviation by player i in phase II_i or III_i is more complex. First, consider phase II_i. It is easy to see that i's payoff to following the equilibrium in this phase is higher the fewer periods of the phase that remain. Once he leaves the phase, his payoff will (approximately) $u_i - \nu > w_i$, while he receives w_i during each period of punishment. Since the punishment for deviation does not depend on how many periods of phase II_i were left when the deviation occurred, obviously, the payoff to deviating is also independent of this. Hence without loss of generality, we can focus on i's incentive to deviate at the beginning of phase II_i.

So suppose i deviated in the preceding period so we are now beginning phase II_i. Recall that i is supposed to change to any maxmin action and remain there for M_i periods, following the cycle for phase III_i afterward. We constructed the cycles so that the cycle for phase III_i does not start with i playing a maxmin action, so he must change actions at the beginning of III_i. Hence if i follows the equilibrium strategies from this point, his payoff will be at least

$$-\varepsilon_n + (1-\delta_n)\sum_{t=0}^{M_i-1} \delta_n^t w_i + \delta_n^{M_i}\big[u_i^i(n) - \varepsilon_n\big].$$

Note that i might not incur a switching cost if the deviation was to a maxmin action. Also, recall that in the first period of punishment, i plays one of his maxmin actions, but the other players will be choosing actions which minimize his payoff against his deviation action. Hence his payoff in that period must be at least w_i. Of course, in the remainder of the phase, his payoff will be exactly w_i each period. Let

$$\hat{u}_i(n) = (1-\delta_n)\sum_{t=1}^{M_i-1} \delta_n^t w_i + \delta_n^{M_1}\big[u_i^i(n) - \varepsilon_n\big],$$

so we can write our lower bound on i's payoff from following the equilibrium as

$$-\varepsilon_n + (1-\delta_n)w_i + \hat{u}_i(n). \tag{4}$$

For the deviation payoff, consider two cases. First, suppose the deviation involves not changing actions in the first period. Note that this is only a deviation if i did not play a maxmin action in the preceding period. Recall that the other players are choosing actions which minimize i's payoff to his previous period's action. Hence the payoff i earns in the period of deviation must be less than w_i. After this, following the equilibrium strategies must earn exactly the same payoff as he would have gotten had he followed them from the outset. That is, the deviation simply adds a period with a payoff less than w_i, delaying the equilibrium payoff. Since the equilibrium payoff must be greater than w_i for n sufficiently large, such a deviation cannot be profitable.

Next, suppose the deviation involves changing actions but to some action other than one of the maxmin actions. In this case, the payoff to deviating and then following the equilibrium strategies cannot exceed

$$-\varepsilon_n(1+\delta_n) + (1-\delta_n)(1+\delta_n)\max_{a\in A} u_i(a) + \delta_n\hat{u}_i(n). \tag{5}$$

Hence the deviation is not profitable if the expression in (4) exceeds the above or

$$\hat{u}_i(n) + \delta_n \frac{\varepsilon_n}{1-\delta_n} \geqslant (1+\delta_n)\max_{a\in A} u_i(a) - w_i.$$

This holds for n sufficiently large if

$$u_i - \nu - \eta + k > 2\max_{a\in A} u_i(a) - w_i.$$

Because $u_i - \nu - \eta > w_i$, a sufficient condition is $k > 2[\max_{a\in A} u_i(a) - w_i]$ which holds by our assumption on k. Hence for n sufficiently large, again, the deviation is not profitable.

Finally, consider a deviation by player i in phase III_i. Suppose we are at the beginning of the ℓth period of the phase III_i cycle. Obviously, i's payoff over the first $\ell - 1$ periods cannot exceed

$$(1-\delta_n)\sum_{t=0}^{\ell-2}\delta_n^t \max_{a\in A} u_i(a).$$

Hence i's continuation payoff over the rest of the cycle cannot be less than

$$\frac{1}{\delta_n^{\ell-1}}\left[\left(1-\delta_n^{L_i}\right)u_i^i(n) - (1-\delta_n)\sum_{t=0}^{\ell-2}\delta_n^t \max_{a\in A} u_i(a)\right].$$

So i's continuation payoff to following the equilibrium strategy must be at least

$$\frac{1}{\delta_n^{\ell-1}}\left[\left(1-\delta_n^{L_i}\right)u_i^i(n) - (1-\delta_n)\sum_{t=0}^{\ell-2}\delta_n^t \max_{a\in A} u_i(a)\right] + \delta_n^{L_i-\ell+1}u_i^i(n).$$

If i deviates, his payoff is less than or equal to

$$(1-\delta_n)(1+\delta_n)\max_{a\in A} u_i(a) + (1-\delta_n)\sum_{t=2}^{M_i}\delta_n^t w_i + \delta_n^{M_i+1}u_i^i(n).$$

Hence i does not have a profitable deviation if

$$\left[\frac{1}{\delta_n^{\ell-1}}\,\frac{1-\delta_n^{L_i}}{1-\delta_n} + \frac{\delta_n^{L_i-\ell+1}-\delta_n^{M_i+1}}{1-\delta_n}\right]u_i^i(n) \geqslant \frac{1}{\delta_n^{\ell-1}}\sum_{t=0}^{\ell-2}\delta_n^t \max_{a\in A} u_i(a) + (1+\delta_n)\max_{a\in A} u_i(a) + \sum_{t=2}^{M_i}\delta_n^t w_i.$$

This holds for large enough n if

$$u_i - \nu - \eta > \frac{\ell+1}{M_i+\ell}\max_{a\in A} u_i(a) + \frac{M_i-1}{M_i+\ell}\, w_i.$$

Since the right-hand side is increasing in ℓ, this holds for all $\ell = 1,\ldots,L_i$ iff

$$u_i - \nu - \eta > \frac{L_i+1}{M_i+L_i}\max_{a\in A} u_i(a) + \frac{M_i-1}{M_i+L_i}\, w_i$$

which is required by our definition of M_i. Hence for all n sufficiently large, i does not have a profitable deviation.

Summarizing, then, we have a finite set of inequalities such that these strategies form a subgame perfect equilibrium iff each inequality holds.[16] For each, there is an $\bar{n}$ such that the inequality holds for all $n \geqslant \bar{n}$. Hence for every n larger than the largest of these finitely many $\bar{n}$'s, these strategies form a subgame perfect equilibrium. □

Appendix B. Proof of Theorem 3

This result is a corollary to Theorem 4. To see this, fix any payoff $u \in F^* \cap W$. Fix a sequence of strictly positive numbers η_ℓ converging to 0. For each η_ℓ, Theorem 4 establishes that there is a k_ℓ and a $\bar{u}(\ell)$ within η_ℓ of u such that

$$\bar{u}(\ell) \in \lim_{\varepsilon\downarrow 0,\delta\uparrow 1}{}^{k_\ell}\ \mathcal{U}(\varepsilon,\delta).$$

In other words, for each ℓ, there is a sequence $(\varepsilon_n^\ell, \delta_n^\ell)$ converging to $(0,1)$ with $\varepsilon_n/(1-\delta_n) \to \bar{k}_\ell$ and a sequence $u^n(\ell)$ such that $u^n(\ell) \in \mathcal{U}(\varepsilon_n^\ell, \delta_n^\ell)$ and $u^n(\ell)$ converges to $\bar{u}(\ell)$ within η_ℓ of u.

16 It is not hard to use that the fact that finite cycles are used in phases I and each III_i and that the punishment in each phase II_i is finite to show that there are only finitely many inequalities to deal with.

So define a new sequence $(\hat{\varepsilon}_n, \hat{\delta}_n)$ as follows. For each n, choose any m_n such that $\max_i |u_i^{m_n}(n) - \bar{u}_i(n)| < 1/n$. Let $(\hat{\varepsilon}_n, \hat{\delta}_n) = (\varepsilon_n^{m_n}, \delta_n^{m_n})$. Let $\hat{u}^n = u^{m_n}(n)$. Then $(\hat{\varepsilon}_n, \hat{\delta}_n)$ converges to $(0, 1)$, $\hat{\varepsilon}_n/(1-\hat{\delta}_n)$ converges to infinity, and $\hat{u}^n$ converges to u. Hence

$$u \in \lim_{\varepsilon\downarrow 0, \delta\uparrow 1}{}^{\infty}\ \mathcal{U}(\varepsilon, \delta). \qquad \square$$

Appendix C. Proof of Theorem 5

Lemma 6. *Fix any $\eta > 0$ and any k. Then for all $u = (u_1, \ldots, u_I) \in F \cap W$, there is a finite cycle of action profiles such that for any sequence $(\varepsilon_n, \delta_n) \to (0, 1)$ with $\varepsilon_n/(1-\delta_n) \to k$, the payoff to i along this sequence converges as $n \to \infty$ to within η of u_i.*

Proof. The key difference between this lemma and Lemma 5 is that we require $u \in F \cap W$ instead of $F^* \cap W$ and allow for arbitrary k. To see that this is possible, fix any η and any k. Fix any $u \in F \cap W$. By definition, $u \in F$ implies that there exists a probability distribution α over A such that

$$u_i = \sum_{a\in A} \alpha(a) u_i(a).$$

It is easy to see that we can find integers $c^a \geqslant 0$ for all $a \in A$ such that

$$\max_i \left| \sum_{a\in A} \left(\alpha(a) - \frac{c^a}{C} \right) u_i(a) \right| < \frac{\eta}{2}$$

where $C = \sum_{a\in A} c^a$. Choose any integer L such that

$$L > \frac{2k\mathcal{A}}{\eta C},$$

where $\mathcal{A}$ is the cardinality of A. In other words,

$$k\frac{\mathcal{A}}{LC} < \frac{\eta}{2}.$$

Construct a cycle as follows. Fix any order over the action profiles a which have $c^a > 0$. The players then play the action profiles in this sequence where profile a is played for $c^a L$ periods. After this, the cycle starts over. The length of the cycle, then, is $\sum_a c^a L = CL$. Hence the relative frequency with which a is played is $c^a L/CL = c^a/C$. We cannot say exactly how many times player i changes actions over the course of the cycle. However, it is certainly fewer than $\mathcal{A}$ times since this is the number of action profiles. Let Z_i be the number of times i changes actions. Obviously, i's payoff over this infinite sequence of actions converges as $n \to \infty$ to

$$\sum_{a\in A} \frac{c^a}{C} u_i(a) - k\frac{Z_i}{CL}.$$

So

$$\max_i \left| u_i - \left[\sum_{a\in A} \frac{c^a}{C} u_i(a) - k\frac{Z_i}{CL} \right] \right| \leqslant \max_i \left| \sum_{a\in A} \left(\alpha(a) - \frac{c^a}{C} \right) u_i(a) \right| + k\frac{\mathcal{A}}{LC} < \frac{\eta}{2} + \frac{\eta}{2} = \eta. \qquad \square$$

To complete the proof, simply construct strategies exactly as in the proof of Theorem 4. The only condition on k used in that part of the proof of Theorem 4 was $k > 2\max_i[\max_{a\in A} u_i(a) - w_i]$, which holds by assumption. $\square$

Appendix D. Proof of Theorem 6

We first show that $c(U_{\geqslant}) \cap W \subseteq \lim_{\varepsilon\downarrow 0} \mathcal{U}_\infty(\varepsilon)$. So fix any $\hat{u} \in c(U_{\geqslant}) \cap W$. By rewardability and convexity of $c(U_{\geqslant}) \cap W$, there must be a $\hat{u}'$ arbitrarily close to $\hat{u}$ with $\hat{u}'_i > w_i$. Hence we may as well assume that $\hat{u} \gg w$. By definition, there are action profiles, say $a^1, \ldots, a^Z$, and strictly positive numbers $\alpha_1, \ldots, \alpha_Z$ such that $u(a^z) \geqslant w$ for all z, $\sum_{z=1}^{Z} \alpha_z = 1$, and $\hat{u} \leqslant \sum_z \alpha_z u(a^z)$. Let $\bar{u} = \sum_z \alpha_z u(a^z)$ and let $x_i = \bar{u}_i - \hat{u}_i$.

By genericity, each player i has a unique maxmin action which we denote $\bar{a}_i$.

Fix an $\varepsilon > 0$ such that $4\varepsilon < \min_i \hat{u}_i - w_i$ and

$$2\varepsilon < \min_i \min_{u,u' \in u(A) | u_i \neq u'_i} |u_i - u'_i|. \tag{6}$$

Rewardability implies that there is no player whose payoff is constant over all $u \in u(A)$. Hence the right-hand side is strictly positive, so this is possible.

For each i, let c_i denote the largest integer such that $c_i \varepsilon \leqslant x_i$. Since $x_i = \bar{u}_i - \hat{u}_i \geqslant 0$, $c_i \geqslant 0$. Let $u^*_i = \bar{u}_i - c_i\varepsilon$ evaluated at this largest c_i. Note that c_i and thus u^*_i are functions of ε, though we omit this dependence in the notation. Clearly, as $\varepsilon \downarrow 0$, $u^* \to \hat{u}$. Also, $u^*_i - \varepsilon \geqslant \hat{u}_i - \varepsilon > w_i$ by our assumption that $\varepsilon < 4\min_i(\hat{u}_i - w_i)$. We now show that there is a payoff vector u' satisfying $u^*_i - \varepsilon \leqslant u'_i \leqslant u^*_i$ such that $u' \in \mathcal{U}_\infty(\varepsilon)$ for all sufficiently small ε. Since both $u^*_i - \varepsilon$ and u^*_i converge to $\hat{u}_i$ as $\varepsilon \downarrow 0$, this demonstrates that $\hat{u} \in \lim_{\varepsilon\downarrow 0} \mathcal{U}_\infty(\varepsilon)$.

To show this, construct strategies as follows. Play begins in phase I and remains there as long as there are no deviations. In the first period of this phase, if c_i is even (where 0 is treated as even), player i plays his maxmin action, $\bar{a}_i$. Otherwise, he plays any other action. For the next several periods, each player changes between $\bar{a}_i$ and any other action, concluding when he has changed actions c_i times. At this point, by construction, he will be back to his maxmin action, $\bar{a}_i$. Once all players have completed these changes, they observe the outcome of a public randomization which selects a^z with probability α_z. Once the outcome of the public randomization is revealed, each player i plays a^z_i. No player changes actions again, so a^z is played forever after. Let u'_i denote i's payoff if this path is followed. It is easy to see that $u^*_i - \varepsilon \leqslant u'_i \leqslant u^*_i$.

If player i deviates by playing action a_i, we move to phase II(i, a_i). At the beginning of phase II(i, a_i), all players other than i change actions to the action profile which minimizes i's payoff to action a_i. The players other than i remain at these actions while i changes actions a random number of times which is characterized below. The number of changes is computed so that i ends up at his maxmin action. In the period where he carries out this last change, each player $j \neq i$ also plays his maxmin action, $\bar{a}_j$. After this, we again have a public randomization to choose an action profile where a^z is chosen with probability α_z. If a^z is chosen, player j plays a^z_j, switching to this action if need be, and actions are never changed again. We remain in this phase as long as there are no deviations from it, moving to phase II(j, a_j) if j deviates by playing a_j, etc. We assign an arbitrary subgame perfect continuation if there are multiple simultaneous deviations in any phase.

The number of times i must switch actions in phase II(i, a_i) is computed as follows. Let

$$p_i = \sum_{z | a^z_i \neq \bar{a}_i} \alpha_z.$$

In other words, p_i is the probability that i must change actions from his maxmin action $\bar{a}_i$ when the outcome of the public randomization is revealed. Let d_i be the largest integer satisfying

$$\bar{u}_i - d_i\varepsilon - p_i\varepsilon \geqslant w_i.$$

We construct a randomization such that d_i is the expected number of times i changes actions and i is sure to end his changes at his maxmin action, $\bar{a}_i$.[17] Specifically, if $a_i = \bar{a}_i$ and d_i is even or if $a_i \neq \bar{a}_i$ and d_i is odd, then i changes actions between $\bar{a}_i$ and any other action exactly d_i times. Note that he will end up at $\bar{a}_i$ in either case. Otherwise, he changes actions $d_i - 1$ times, ending up at $\bar{a}_i$. We then have a public randomization to determine whether i needs to change actions again. With probability 1/2, he stops, while with probability 1/2, he changes actions two more times. It is easy to see that i changes actions d_i times in expectation, ending at $\bar{a}_i$.

To see that these strategies form a subgame perfect equilibrium, first, consider i's incentive to deviate in phase II(i, a_i) before the outcome of the public randomization determining a is revealed. Clearly, i's payoff to following the equilibrium is at least $\bar{u}_i - d_i\varepsilon - p_i\varepsilon$ (higher if he has already carried out some of the necessary changes of action). By construction, this is at least w_i. If i deviates, he can either deviate and return to the equilibrium strategies (generating the same payoff or less), change actions infinitely often (yielding a payoff of $-\infty$), or change actions some finite number of times and then never change again (yielding w_i at most). Hence he cannot gain by deviating.

[17] If i has three or more actions, this is easy to do with pure strategies. The construction here shows that there is no difficulty even if i has only two actions.

Next, consider i's incentive to deviate immediately after the outcome of the public randomization is revealed in any phase. Suppose the outcome of the public randomization is revealed to be a^z. If $u_i(a^z) > w_i$, then i must change actions, so his payoff to following the equilibrium is $u_i(a^z) - \varepsilon$. By Eq. (6), it must be true that $2\varepsilon < u_i(a^z) - w_i$, so $u_i(a^z) - \varepsilon > w_i + \varepsilon$. Suppose i deviates but follows the equilibrium thereafter. Then from the definition of d_i, we know that his payoff will be less than $w_i + \varepsilon$. Hence he cannot gain from such a deviation. If he deviates and continues to deviate, as seen above, his payoff cannot exceed w_i, so, again, he cannot gain.

So suppose $u_i(a^z) = w_i$. In this case, i does not have to change actions, so his payoff to following the equilibrium is w_i. If he deviates, he must change actions. Hence if he deviates and then follows the equilibrium thereafter, his payoff is $\bar{u}_i - (d_i + 1)\varepsilon - p_i\varepsilon < w_i$, where the inequality follows the definition of d_i. As discussed above, if he deviates and continues to deviate, his payoff will be less than $w_i - \varepsilon$. Hence, again, he cannot gain from deviating.

Next, consider any agent $j \neq i$ in any phase $\mathrm{II}(i, a_i)$. By following the equilibrium, j's payoff is at least $\bar{u}_j - 3\varepsilon$ (one change to punish i, one to go to his maxmin action before the outcome of the public randomization is revealed, and one to change to the appropriate action after that). By assumption, $4\varepsilon < \bar{u}_j - w_j$, so $\bar{u}_j - 3\varepsilon > w_j + \varepsilon$. It is easy to see that if j deviates, his payoff will be less than $w_j + \varepsilon$, so this implies j has no incentive to deviate.

Essentially the same argument applies to a deviation by any agent in phase I. Hence these strategies form a subgame perfect equilibrium for any sufficiently small $\varepsilon > 0$.

This demonstrates that $c(U_\geqslant) \cap W \subseteq \lim_{\varepsilon \downarrow 0} \mathcal{U}_\infty(\varepsilon)$. To complete the proof, then, suppose $u \notin c(U_\geqslant) \cap W$. We now show that there is no equilibrium payoff nearby. Since $c(U_\geqslant) \cap W$ is closed and does not contain u, for every sufficiently small $\varepsilon > 0$ and every u' within ε of u, we have $u' \notin c(U_\geqslant) \cap W$. Choose any such ε which is small enough that

$$\varepsilon < w_i - u_i(a)$$

for all a and i such that $u_i(a) < w_i$. Suppose, contrary to our claim, that there is a u' within ε of u with $u' \in \mathcal{U}_\infty(\varepsilon)$. Fix such a u' and the equilibrium generating this payoff.

Obviously, we must have $u' \in W$ as any player i can guarantee himself a payoff of w_i from a constant action. Hence the fact that $u' \notin c(U_\geqslant) \cap W$ implies that $u' \notin c(U_\geqslant)$. Hence there is some action profile $\hat{a}$ with $u_i(\hat{a}) < w_i$ which is played infinitely often with strictly positive probability.

Let $\mathcal{P}$ denote the set of all possible paths (infinite sequences of action profiles). Let μ denote the probability distribution over $\mathcal{P}$ induced by the equilibrium generating the payoff u' (including the effect of the public randomizations if strategies are based on these). For any finite T, let $\mathcal{P}^*_T(a)$ denote the set of paths in $\mathcal{P}$ which have absorbed at action profile a by date T—that is,

$$\mathcal{P}^*_T(a) = \{(a^1, a^2, \ldots) \in \mathcal{P} \mid a^t = a,\ \forall t \geqslant T\}.$$

Let $\mathcal{P}^*(a)$ denote the set of paths that eventually absorb at action profile a—that is,

$$\mathcal{P}^*(a) = \bigcup_{T=0}^{\infty} \mathcal{P}^*_T(a).$$

Similarly, let $\mathcal{P}_d$ denote the set of paths in $\mathcal{P}$ which do not absorb—that is, $\mathcal{P}_d = \mathcal{P} \setminus \bigcup_{a \in A} \mathcal{P}^*(a)$. It is not hard to see that $\mu(\mathcal{P}_d) = 0$. If this were not zero, then the expected total switching costs of the players would necessarily be infinite, meaning that some player's payoff is $-\infty$, so obviously his strategy cannot be optimal.

Our selection of $\hat{a}$ implies that $\mu(\mathcal{P}^*(\hat{a})) > 0$. Recall that we chose $\hat{a}$ to be some profile played infinitely often with strictly positive probability. Since no path in $\mathcal{P}^*(a)$ for $a \neq \hat{a}$ has this property and since $\sum_{a \in A} \mu(\mathcal{P}^*(a)) = 1$, we must have $\mu(\mathcal{P}^*(\hat{a})) > 0$.

Let $\mathcal{P}^t(a)$ denote the set of paths in $\mathcal{P}$ with $a^t = a$. For any $T \leqslant t$ and $a \neq \hat{a}$, we must have $\mathcal{P}^*_T(a) \cap \mathcal{P}^t(\hat{a}) = \emptyset$. That is, if a path absorbs at a at date $T \leqslant t$, then a must be played at t, not $\hat{a}$. Hence for all $a \neq \hat{a}$,

$$\mu(\mathcal{P}^*(a) \cap \mathcal{P}^t(\hat{a})) = \sum_{T \geqslant t+1} \mu(\mathcal{P}^*_T(a) \cap \mathcal{P}^t(\hat{a})).$$

Hence for all $a \neq \hat{a}$,

$$\mu(\mathcal{P}^*(a) \cap \mathcal{P}^t(\hat{a})) \to 0$$

as $t \to \infty$.

In light of this, consider

$$\mu\big(\mathcal{P}^*(\hat{a}) \mid \mathcal{P}^t(\hat{a})\big) = \frac{\mu(\mathcal{P}^*(\hat{a}) \cap \mathcal{P}^t(\hat{a}))}{\mu(\mathcal{P}^t(\hat{a}))}.$$

Since all the $\mathcal{P}^*(a)$ sets are disjoint and their union has probability 1, we can rewrite this as

$$\frac{\mu(\mathcal{P}^*(\hat{a}) \cap \mathcal{P}^t(\hat{a}))}{\sum_{a \in A} \mu(\mathcal{P}^*(a) \cap \mathcal{P}^t(\hat{a}))}.$$

Clearly, as $t \to \infty$, this converges to

$$\frac{\mu(\mathcal{P}^*(\hat{a}))}{\mu(\mathcal{P}^*(\hat{a}))} = 1.$$

(Note that this is well defined since $\mu(\mathcal{P}^*(\hat{a})) > 0$.) In short, $\mu(\mathcal{P}^*(\hat{a}) \mid \mathcal{P}^t(\hat{a})) \to 1$ as $t \to \infty$.

Consider player i, the player for whom $u_i(\hat{a}) < w_i$. Fix any t for which $\hat{a}$ is played at t with positive probability in the equilibrium. Consider the following strategy for player i: follow the equilibrium strategy until the equilibrium strategies call for $\hat{a}$ to be played at period t. Then deviate to one of the maxmin actions forever after. Clearly, since the original strategies form an equilibrium, this alternative strategy cannot be better for i for any choice of t. In comparing i's payoff in the equilibrium to i's payoff from the alternative strategy, obviously, we can condition on the set of paths for which $\hat{a}$ is played at time t—for any other paths, the payoff difference is zero. If player i deviates at time t as specified, his expected payoff from that point onward is at least $w_i - \varepsilon$. But as $t \to \infty$, player i's expected continuation payoff if he does not deviate is converging to $u_i(\hat{a})$. Since $u_i(\hat{a}) < w_i - \varepsilon$, there is some large t for which i strictly prefers the deviation, a contradiction. □

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