An Elementary Proof of the Optimality of Threshold Mechanisms

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In this short note, I give an elementary proof of a key result in Ben-Porath, Dekel, and Lipman (2014), henceforth referred to as BDL.

I first state the key assumptions. We have independent random variables, $t_1, \ldots, t_I$. The support of t_i, T_i, is a subset of the interval $[\underline{t}_i, \bar{t}_i]$ where we assume $-\infty < \underline{t}_i < \bar{t}_i < \infty$. The cdf of t_i is denoted F_i and we assume it has a density f_i. We assume $F_i(0) < 1$ for at least one i — otherwise, the analysis is trivial.

We have exogenously given scalars, $\varphi_1, \ldots, \varphi_I$ where $\sum_i \varphi_i \leq 1$ and $\varphi_i \geq 0$ for all i.

A *mechanism* is a profile of functions $p = (p_1, \ldots, p_I)$ such that $p_i : T \to [0, 1]$ and $\sum_i p_i(t) \leq 1$ for all t. The principal's problem[1] is to choose a mechanism to maximize

$$\sum_i \mathrm{E}_t[p_i(t)t_i]$$

subject to $\hat{p}_i(t_i) \geq \varphi_i$ for all $t_i \in T_i$ and all i where

$$\hat{p}_i(t_i) = \mathrm{E}_{t_{-i}} p_i(t).$$

[1] BDL write this problem in a different but equivalent way. They write the principal's objective function as $\sum_i \mathrm{E}_t[p_i(t)(t_i - c_i)]$ for constants c_i. Our t_i should be interpreted as their $t_i - c_i$.

We say that p is a *threshold mechanism* if there exists $t^* \geq 0$ with the following properties up to sets of measure zero. First, if $t_i < t^*$, then $\hat{p}_i(t_i) = \varphi_i$. Second if t has the property that $t_i > t^*$ for some i, then $p_j(t) = 1$ for that j such that $t_j = \max_k t_k$. Third, either $t^* = 0$ or $\sum_i p_i(t) = 1$ for all t.

This is a slightly different definition than that of BDL. They do not include the third property or the requirement that $t^* \geq 0$. On the other hand, they do prove that their optimal threshold mechanisms satisfy these properties.

Lemma 1. *A threshold mechanism exists.*

Proof. First, suppose $\varphi_i = 0$ for all i. Then the mechanism that gives the good to the agent with the highest type if this type is positive and to no agent otherwise is a threshold mechanism with threshold 0. Henceforth, assume there exists i with $\varphi_i > 0$.

Consider the function

$$g(t^*) \equiv \sum_{i|\varphi_i>0} \frac{\varphi_i}{\prod_{j\neq i} F_j(t^*)}.$$

Clearly, g is continuous and decreasing. Also,

$$g(\max_i \bar{t}_i) = \sum_i \varphi_i \leq 1.$$

Suppose $g(0) \leq 1$. Then consider the following mechanism. If any $t_i > 0$, we allocate the good to the agent with the highest type. If all $t_i < 0$, i receives the good with probability $\varphi_i / \prod_{j\neq i} F_j(0)$. This mechanism is clearly feasible and is a threshold mechanism with $t^* = 0$.

To be more specific, first, suppose $F_j(0) > 0$ for all j. Then

$$\sum_i \frac{\varphi_i}{\prod_{j\neq i} F_j(0)} \leq 1,$$

so we can allocate the good to i with probability $\varphi_i / \prod_{j\neq i} F_j(0)$ and get a feasible allocation. Second, suppose $F_j(0) = 0$ for some j. Since $g(0) < \infty$, we must have $\varphi_i = 0$ for all $j \neq i$. Since there is some agent k with $\varphi_k > 0$, we must have $\varphi_j > 0$. Again, $g(0) < \infty$ then implies $F_i(0) > 0$ for all $i \neq j$. Hence

$$g(0) = \frac{\varphi_j}{\prod_{i\neq j} F_i(0)} \leq 1.$$

So we can give the good to j with probability $g(0)$ when every other agent has a type below 0. Either way, we get a threshold mechanism with threshold 0.

So suppose $g(0) > 1$. By continuity, there exists $t^* > 0$ such that $g(t^*) = 1$. Fix such a t^*. Define the mechanism by giving the good to the agent with the highest type for any t such that $t_i > t^*$ for some i. When $t_i < t^*$ for all i, give the good to i with probability $\varphi_i / \prod_{j \neq i} F_j(t^*)$. This is a threshold mechanism with threshold t^*. ∎

Theorem 1. *A mechanism is optimal if and only if it is a threshold mechanism.*

Proof. First, we show that any threshold mechanism is optimal. Suppose not. Fix a threshold mechanism p which is not optimal. Let p^* denote a feasible mechanism with a strictly higher payoff for the principal.

First, construct a new mechanism, p^{**}, by

$$p_i^{**}(t) = \begin{cases} 0, & \text{if } t_i > t^* \text{ and } \exists j \neq i \text{ with } t_j > t_i \\ p_i^*(t), & \text{if } t_i < t^* \\ p_i^*(t) + \sum_{\{j \mid t_j > t^* \text{ and } t_i > t_j\}} p_j^*(t), & \text{otherwise.} \end{cases}$$

In other words, we move all probability allocated to an agent who is above the threshold and is not the highest type to the highest type. Clearly, $p_i^{**}(t) \geq 0$ for all i and all t and $\sum_i p_i^{**}(t) = \sum_i p_i^*(t) \leq 1$, so p^{**} is a well–defined mechanism. Also, unless $p^{**}(t) = p^*(t)$ except on sets of measure zero, the principal strictly prefers p^{**} to p^*.

There are two important points to note about this mechanism. First, it need not be feasible given φ. That is, we may well have $\hat{p}_i^{**}(t_i) < \varphi_i$ for some t_i. However, this won't affect the argument. The point is that if there is a feasible p^* which is strictly better than p, then this potentially infeasible p^{**} is also strictly better. We will derive a contradiction to the latter, completing the proof of this part of the Theorem.

Second, for all $t_i > t^*$, we must have $\hat{p}_i^{**}(t_i) \leq \hat{p}_i(t_i)$. To see this, note that under p, any $t_i > t^*$ receives the good at profile t if and only if he has the highest type. Under p^{**}, a *necessary* condition for such t_i to receive the good is that he has the highest type, but this might not be sufficient.

Since p^{**} yields the principal at least as high a payoff as p^* and p^* yields a strictly higher payoff than p, we have

$$\sum_i \int_{\underline{t}_i}^{\bar{t}_i} \hat{p}_i^{**}(t_i) t_i f_i(t_i)\, dt_i > \sum_i \int_{\underline{t}_i}^{\bar{t}_i} \hat{p}_i(t_i) t_i f_i(t_i)\, dt_i$$

or

$$\sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^{**}(t_i) - \hat{p}_i(t_i)] t_i f_i(t_i)\, dt_i > \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^{**}(t_i)] t_i f_i(t_i)\, dt_i.$$

When $t^* > t_i$, we have $\hat{p}_i(t_i) = \varphi_i \leq \hat{p}_i^{**}(t_i)$. Hence

$$t^* \sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^{**}(t_i) - \hat{p}_i(t_i)] f_i(t_i)\, dt_i \geq \sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^{**}(t_i) - \hat{p}_i(t_i)] t_i f_i(t_i)\, dt_i.$$

For $t_i > t^*$, we have $\hat{p}_i(t_i) \geq \hat{p}_i^{**}(t_i)$, implying

$$\sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^{**}(t_i)] t_i f_i(t_i)\, dt_i \geq t^* \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^{**}(t_i)] f_i(t_i)\, dt_i$$

Combining these three inequalities gives

$$\begin{aligned} t^* \sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^{**}(t_i) - \hat{p}_i(t_i)] f_i(t_i)\, dt_i &\geq \sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^{**}(t_i) - \hat{p}_i(t_i)] t_i f_i(t_i)\, dt_i \\ &> \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^{**}(t_i)] t_i f_i(t_i)\, dt_i \\ &\geq t^* \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^{**}(t_i)] f_i(t_i)\, dt_i. \end{aligned}$$

Recall that $t^* \geq 0$. Clearly, if $t^* = 0$, this equation gives a contradiction. So $t^* > 0$. Hence

$$\sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^{**}(t_i) - \hat{p}_i(t_i)] f_i(t_i)\, dt_i > \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^{**}(t_i)] f_i(t_i)\, dt_i$$

or

$$\sum_i \int_{\underline{t}_i}^{\bar{t}_i} \hat{p}_i^{**}(t_i) f_i(t_i)\, dt_i > \sum_i \int_{\underline{t}_i}^{\bar{t}_i} \hat{p}_i(t_i) f_i(t_i)\, dt_i.$$

Rewriting,

$$\sum_i \int p_i^{**}(t) f(t)\, dt > \sum_i \int p_i(t) f(t)\, dt$$

But recall that if $t^* > 0$, we have $\sum_i p_i(t) = 1$ for all t. Hence

$$\int \sum_i p_i^{**}(t) f(t)\, dt > \int \sum_i p_i(t) f(t)\, dt = \int f(t)\, dt = 1.$$

Clearly, this is impossible. Hence p must be optimal.

Next, we show that every optimal mechanism is a threshold mechanism. So let p be a threshold mechanism with threshold t^* and let p^* be any other optimal mechanism.

Suppose $\hat{p}_i^*(t_i) > \hat{p}_i(t_i)$ for a positive measure set of $t_i > t^*$. Since under p, any $t_i > t^*$ receives the object if and only if he has the highest type, this implies that there is positive measure set of t_{-i} such that t_i receives the object even though he does not have the highest type. But then we can improve p^* by moving some probability from t_i to the agent with the highest type on such profiles t. Since $\hat{p}_i^*(t_i) > \hat{p}_i(t_i) \geq \varphi_i$, moving a small enough amount of probability this way is feasible. Since this would give a mechanism strictly better for the principal than p^* and p^* is optimal, we know this is impossible. Hence, up to sets of measure zero, $\hat{p}_i^*(t_i) \leq \hat{p}_i(t_i)$ for all $t_i > t^*$.

Also, we know that for $t_i < t^*$, $\hat{p}_i(t_i) = \varphi_i \leq \hat{p}_i^*(t_i)$. Because p and p^* are both optimal,

$$\sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^*(t_i) - \hat{p}_i(t_i)] t_i f_i(t_i)\, dt_i = \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^*(t_i)] t_i f_i(t_i)\, dt_i.$$

Following similar reasoning to the first part of the proof, we see that

$$\begin{aligned}
t^* \sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^*(t_i) - \hat{p}_i(t_i)] f_i(t_i)\, dt_i &\geq \sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^*(t_i) - \hat{p}_i(t_i)] t_i f_i(t_i)\, dt_i \\
&\geq \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^*(t_i)] t_i f_i(t_i)\, dt_i \\
&\geq t^* \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^*(t_i)] f_i(t_i)\, dt_i.
\end{aligned}$$

with at least one strict inequality if and only if $\hat{p}_i^*(t_i) \neq \hat{p}_i(t_i)$ on a positive measure subset of T_i for some i. Recall that $t^* \geq 0$. If $t^* = 0$, we cannot have a strict inequality, so either $t^* > 0$ or $\hat{p}_i^*(t_i) = \hat{p}_i(t_i)$ up to a set of measure zero for all i. Assuming the former, then, we have

$$\sum_i \int_{\underline{t}_i}^{t^*} [\hat{p}_i^*(t_i) - \hat{p}_i(t_i)] f_i(t_i)\, dt_i \geq \sum_i \int_{t^*}^{\bar{t}_i} [\hat{p}_i(t_i) - \hat{p}_i^*(t_i)] f_i(t_i)\, dt_i$$

with strict inequality if and only if $\hat{p}_i^*(t_i) \neq \hat{p}_i(t_i)$ on a positive measure subset of T_i for some i. Rewriting as in the first part of the proof, this implies

$$\int \sum_i p_i^*(t) f(t)\, dt \geq \int \sum_i p_i(t) f(t)\, dt = \int f(t)\, dt = 1,$$

with strict inequality iff $\hat{p}_i^*(t_i) \neq \hat{p}_i(t_i)$ on a positive measure subset of T_i for some i. Since a strict inequality is impossible, we see that $\hat{p}_i^*(t_i) = \hat{p}_i(t_i)$ with probability 1 for all i.

Clearly, then, up to sets of measure zero, we have $\hat{p}_i^*(t_i) = \varphi_i$ for all $t_i < t^*$. For $t_i > t^*$, suppose $\hat{p}_i^*(t_i) = \hat{p}_i(t_i)$, but that the allocation for some profiles t_{-i} differ across p and p^* given t_i. We know that it cannot be the case that t_i receives the good when there is a higher type since we could improve in that case. Hence t_i can only get the good when he has the highest type. But his overall probability of getting the good must exactly equal the probability he has the highest type, so this implies that any deviations from getting the good when he has the highest type are measure zero. Hence, p^* is a threshold mechanism with threshold t^*. ▌

References

[1] Ben-Porath, E., E. Dekel, and B. Lipman, "Optimal Allocation with Costly Verification," *American Economic Review*, **104**, December 2014, 3779–3813.