The $k^{-1}$ scaling of air temperature layer flows

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A number of atmospheric surface layer (ASL) experiments reported a $k^{-1}$ scaling in air temperature spectra $E_{TT}(k)$ at low wavenumber $k$ but other experiments did not. Occurrence of this scaling law in $E_{TT}(k)$ in an idealized ASL flow across a wide range of atmospheric stability regimes is investigated theoretically and experimentally using measurements collected above a lake and a grass surface. Experiments reveal a $k^{-1}$ scaling persisted across different atmospheric stability parameter values ($\xi$) ranging from mildly unstable to mildly stable conditions ($-0.1 < \xi < 0.2$). As instability increases, the $k^{-1}$ scaling vanishes. Based on simplified spectral and co-spectral budgets and using a Heisenberg eddy viscosity as a closure to the spectral flux transfer term, conditions promoting a $k^{-1}$ scaling in $E_{TT}(k)$ are identified. Existence of a $k^{-1}$ scaling is shown to be primarily linked to an imbalance between the production and dissipation rates of half the temperature variance. When $-0.1 < \xi < 0.2$, such imbalance exhibits weak dependence on $\xi$ and hence $z$, which is shown to be the main cause for a $-1$ scaling at low $k$. As the atmosphere becomes more unstable, the imbalance determined from experiments here are not significantly affected by $\xi$, thereby negating conditions promoting a $-1$ scaling in $E_{TT}(k)$. The role of the imbalance between the production and dissipation rates of half the temperature variance in controlling the existence of a $-1$ scaling suggests that the $-1$ scaling in $E_{TT}(k)$ does not necessarily concur with the $-1$ scaling in the spectra of longitudinal velocity $E_{uu}(k)$. This finding explains why some ASL experiments reported $k^{-1}$ in $E_{uu}(k)$ but not in $E_{TT}(k)$.

Key Words: atmospheric surface layer; co-spectral budget; Heisenberg's eddy viscosity; spectral budget; temperature spectra; $k^{-1}$ scaling

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1. Introduction

Since the Kansas experiments, the ‘universal’ or ‘$-5/3$’ character of the Kolmogorov scaling in the inertial subrange of the velocity and air temperature spectra received much attention (Kaimal et al., 1972; Wyngaard and Cote, 1972; Kaimal, 1973; Kaimal and Finnigan, 1994). The low-frequency range of these spectra, on the other hand, received disproportionately less attention as this range appears to be non-universal (Kaimal and Finnigan, 1994). The Kansas experiment reported that the spectra of longitudinal ($u$), lateral ($v$), and vertical ($w$) velocity, as well as the spectra of air temperature ($T$) ‘level off’ as the wavenumber $k$ decreases (Kaimal and Finnigan, 1994). This levelling-off is featured as reduction in the pre-multiplied spectrum $kE_{uu}(k)$ with decreasing $k$ (i.e. large scales), where $E_{uu}(k)$ is the spectrum of a flow variable $s = u, v, w, T$, and $k$ is the wavenumber. Other measurements in the laboratory and in the atmospheric surface layer (ASL) show that the spectra of $u (E_{uu}(k))$ can have a distinct $-1$ scaling at low $k$ for near-neutral conditions (Katul and Chu, 1998; Calaf et al., 2013). The $k^{-1}$ scaling in $E_{uu}(k)$ was further conjectured from dimensional analysis (Kader and Yaglom, 1991; Katul et al., 1995), simplified spectral budgets (Tchen, 1953, 1954; Katul et al., 2012; Banerjee and Katul, 2013; Banerjee et al., 2015), and asymptotic matching between outer and inner layers in laboratory boundary layers (Perry and Abell, 1975, 1977; Perry et al., 1986, 1994).

While a $-1$ scaling has also been identified at low $k$ in $E_{TT}(k)$ in some ASL field experiments under near-neutral, mildly unstable, and mildly stable conditions (Kader and Yaglom, 1991; Katul et al., 1995; Li et al., 2015a), the theoretical conditions promoting its onset have not been considered beyond the $z$ independence argument proposed by Kader and Yaglom (1991), where $z$ is the height above the ground surface. In this argument, when the spectrum of an arbitrary flow variable is normalized by...
'inner-layer' or near-neutral ASL similarity theory variables, then

\[ \frac{E_u(k)}{S_{zz}} = f_u(kz), \tag{1} \]

where \( S_u = -\overline{w'\xi}/u_\tau \) is an ASL scaling variable, \( u_\tau \) is the friction velocity, and \( f_u \) is a universal similarity function. Here, the overline denotes Reynolds averaging and primed quantities denote turbulent fluctuations from the Reynolds averages. For \( s = T \), \( E_{TT}(k) \) becomes the air temperature spectrum that satisfies

\[
\int_0^\infty E_{TT}(k') \, dk' = T^2, \tag{2}
\]

and \( T^2 = \sigma_T^2 \) is the air temperature variance. \( T_\tau \) is a characteristic ASL turbulent temperature scale formed by the ratio of the turbulent sensible heat flux \( (\overline{w'\theta}) \) and \( u_\tau \). For eddies commensurate with the boundary-layer height (i.e. \( kz < 1 \)), \( E_u(k) \) becomes independent from \( z \) and this independence can be achieved when \( f_u(kz) \sim (kz)^{-1} \) resulting in \( E_u(k) \sim S_{zz} k^{-1} \) in the near-neutral ASL. In fact, this argument requires that \( E_{TT}(k) \sim T_\tau^2 k^{-1} \) be accompanied by a \( -1 \) power-law scaling in the velocity spectra \( (E_{ww}(k), E_{uw}(k) \text{ and } E_{uw}(k)) \), though early experiments over water do not support such a claim (Pond et al., 1966). Evidently, other mechanisms that control or modulate the existence of the \(-1\) scaling in \( E_{TT}(k) \) must be at play, and they need not be tied to the spectrum of the velocity components (particularly \( E_{ww}(k) \)). Moreover, ASL experiments reporting \( E_{TT}(k) \sim k^{-1} \) suggests that the \(-1\) power law extends to \( kz \sim 1 \), raising questions about the validity of the aforementioned similarity argument. Predicting the onset of a \(-1\) scaling in \( E_{TT}(k) \) and its 'disconnect' from the \(-1\) power-law scaling in velocity (particularly \( E_{ww}(k) \)), and explaining why the \(-1\) scaling in \( E_{TT}(k) \) can extend to \( kz \sim 1 \) when such \(-1\) power-law scaling occurs in \( E_{TT}(k) \) frames the scope of this study.

The specific objectives of this study are two-fold. The first is to experimentally explore whether certain atmospheric stability regimes promote a \(-1\) scaling at low \( k \) in \( E_{TT}(k) \) and what is the wavenumber range over which such power-law scaling holds. The second is to unfold the main causes for the onset of this \(-1\) scaling by employing recently developed spectral and co-spectral budgets (Banerjee and Katul, 2013; Katul et al., 2013; Katul et al., 2014; Banerjee et al., 2015; Li et al., 2015b), where the spectral flux transfer term is ‘closed’ by a Heisenberg’s eddy viscosity concept (Heisenberg, 1948). It is shown that existence of a \(-1\) scaling in \( E_{TT}(k) \) is linked to an imbalance between production and dissipation rates in the temperature variance budget, alluded to in previous studies (Hsieh and Katul, 1997; Zilitinkevich et al., 2013). The article is organized as follows: section 2 presents experiments in the ASL over a lake and a grass surface showing the range of atmospheric stability conditions favouring the onset of a \(-1\) scaling in \( E_{TT}(k) \); section 3 proposes a theory for the onset of a \(-1\) scaling in \( E_{TT}(k) \) using the aforementioned spectral budget; section 4 shows how the model calculations and experimental findings are reconciled, and section 5 concludes the work.

2. Experimental data

Three-dimensional velocity and air temperature time series were sampled at 20 Hz frequency and at four heights (1.65, 2.30, 2.95 and 3.60 m) above an extensive lake surface (Lake Geneva, Switzerland) using four pairs of sonic anemometers (Campbell Scientific Inc., model CSAT3) and open-path gas analyzers (LI-COR, Inc., model LI-7500) from mid August to late October in 2006. The lake, which is one of the largest lakes in Western Europe, is situated on the north side of the Alps (372 m above sea level) and is shared between Switzerland and France. The set-up, data quality controls, and post-processing are presented elsewhere (Vercauteren et al., 2008; Li and Bou-Zeid, 2011; Li et al., 2012). The temperature time series measured by sonic anemometer are corrected for humidity effects (Schotanus et al., 1983) to obtain air temperature time series at each height using the method of Li et al. (2012). Calculations of turbulent fluxes follow standard eddy-covariance approaches (Li and Bou-Zeid, 2011; Li et al., 2012) with an averaging interval of 30 min per run. Data with a friction velocity \( u_\tau < 0.01 \text{ ms}^{-1} \) and sensible heat flux \( \overline{H} < 5 \text{ W m}^{-2} \) were excluded from the analysis. Similar to Li et al. (2015a), runs where measured turbulent momentum, sensible heat, and latent heat fluxes differed by more than 10% at the four heights were excluded from the analysis here to ensure turbulent fluxes were constant with height in the ASL. In addition, the turbulence intensity has to be smaller than 0.5 to justify the use of Taylor’s hypothesis (Stull, 1988) when inferring wavenumbers from time. In addition, similar to Sun et al. (2015), highly non-stationary segments were excluded using the quality control described by Foken and Wichura (1996).

Data collected by a continuous meteorological observatory in Wageningen, the Netherlands (7 m above sea level) were also used here. The surface is covered with perennial ryegrass (Lolium perenne L.) and rough bluegrass (Poa pratensis L.). The grass is mown weekly during the growing season and has a typical mean height of 0.1 m and a leaf area index of 2.9 m² m⁻². The soil is classified as heavy basin clay. Surface heterogeneity is present because of ditches, different grass species in adjacent fields, electricity towers, and a farm some 500 m downwind. Turbulent fluxes of momentum, heat, and mass (H2O and CO2) were measured on a lattice tower instrumented with an eddy-covariance system installed at a height of 3.5 m. This system includes a three-dimensional sonic anemometer (3-D Solent, Gill Instruments Ltd., model A1012R2) and an open-path gas analyzer (LI-COR, Inc., model LI-7500), as well as a locally produced fine-wire thermocouple. The data were quality controlled using the eddy-covariance processing toolbox (Sturm et al., 2012). In addition, data with friction velocity \( u_\tau < 0.05 \text{ ms}^{-1} \), ASL temperature scale \( T_\tau < 0.05 \text{ K} \), or turbulence intensity > 0.5 were excluded from the analysis. To avoid dew conditions, data with relative humidity higher than 90% were also excluded.

For both experiments, spectra are calculated for each 30 min run after linear-defining following the standard Fourier transform method (Stull, 1988) and are smoothed using a periodic Hamming window without overlap. Dissipation rate of half the temperature variance \( N_\tau \) is calculated using a third-order mixed velocity–temperature structure function \( D_{TTu}(r) \) from

\[
N_\tau = -\frac{3}{4} \frac{D_{TTu}(r)}{r},
\]

for each run (Kolmogorov, 1941; Monin and Yaglom, 1971) where

\[
D_{TTu}(r) = |T(x+r) - T(x)|^2 |u(x+r) - u(x)|,
\]

where \( r \) is the separation distance inferred from time series and the mean velocity using Taylor’s frozen turbulence hypothesis (Taylor, 1938), and should lie in the inertial subrange. Analysis similar to those in Katul et al. (1997) (conducted above a grass surface) indicated that \( r = z/2 \) was sufficient for the lake data but \( r = z/5 \) was required for the grass data, where \( z \) is the height above the surface. The \( N_\tau \) is also calculated from the second-order temperature structure function. The two estimates of \( N_\tau \) are comparable, as shown in Appendix A.

The 30 min data runs were then separated into eight stability regimes according to the stability parameter \( \zeta = z/L \), presented in Table 1, where \( L = -\kappa \nu / (\beta \overline{w'\theta}) \) is the Obukhov length (Obukhov, 1946), \( \kappa = 0.4 \) is the von Kármán constant, \( \beta = g/T_v \) is the buoyancy parameter, \( g \) is the gravitational acceleration, and
for the idealized ASL, the spectral flux transfer term can be finite at any height z though its integral over the entire spatial domain (i.e. boundary-layer height here) is negligible. The turbulent transport term describes the transfer of temperature variance across scales in wavenumber space satisfying the normalizing condition

where the integral of the spectral flux transfer term is zero (i.e. \( \int_0^\infty TR(k) dk' = 0 \)) (Hinze, 1959).

Note the difference between the turbulent transport term VT in Eq. (2) and the spectral flux transfer term TR in Eq. (3). The turbulent transport term VT describes the transport of temperature variance in physical space and can be finite at any height z though its integral over the entire spatial domain (i.e. boundary-layer height here) is negligible. The spectral flux transfer term TR describes the transfer of temperature variance across scales in wavenumber space satisfying the normalizing condition \( \int_0^\infty TR(k) dk' = 0 \).

When \( k \to \infty \) for Eq. (4), \( N_a = D_m \int_0^\infty k^2 E_{TT}(k) dk' \), which recovers the definition of \( N_a \). However, when \( k = 0 \) for Eq. (4), \( N_a = -w T_d T/dz \), implying that the budget Eq. (2) is in equilibrium, namely, the production rate \(-w T_d T/dz\) is in balance with the dissipation rate \( (N_a) \).

This further implies that Eq. (3) (and hence Eq. (4)) is not a simple transformation of Eq. (2). To account for possible imbalances between the production and the dissipation rates of half the temperature variance (i.e. VT) at an arbitrary height z, the \( N_a \) on the left-hand side of Eq. (4) must be adjusted by VT so that when \( k = 0 \), \( N_a = -w T_d T/dz + VT \).

For ASL flows, the focus is on \( k = z^{-1} \), namely, when the eddy size is set to the distance from the wall (Townsend, 1976), and Eq. (4) at \( k = z^{-1} \) becomes:

which is to be evaluated after discussing the parametrization for the production term, the spectral flux transfer term and the molecular dissipation term.

### 3.3. Parametrization for the production term

To evaluate Eq. (5), it is evident that the spectral shape of \( F_{wT}(k) \) in the range of \([z^{-1}, \infty]\) must be determined. For simplicity, it is assumed that \( F_{wT}(k) \) follows its classical \(-7/3\) scaling to be consistent with previous co-spectral budget models (Kutal et al., 2013, 2014; Li et al., 2015b) and empirical and numerical studies (Lumley, 1967; Kaimal and Finnigan, 1994; Bos et al., 2004; Bos and Bertoglio, 2007), given by:

\[
F_{wT}(k) = C_{wT} \frac{dT}{dz} k^{1/3} k^{-7/3},
\]
where $\epsilon$ is the mean turbulent kinetic energy dissipation rate and 
\[ C_{wT} = QC_{wT} = \left[ 1 - \frac{1}{\frac{C_T}{C_T} \frac{\zeta}{\zeta} \frac{C_{\phi m}(\zeta)}{C_{\phi m}}} \right] C_{wT}. \] (7) 

$C_{wT}$ ($\approx 0.15$) is the co-spectral similarity constant for near-neutral conditions (Kaimal and Finnigan, 1994) and $Q$ represents the impact of atmospheric stability on the co-spectral similarity constant (Katul et al., 2013; 2014; Li et al., 2015b). $C_T$ ($\approx 0.6$) is a constant associated with isotropization of the production term in the sensible heat flux budget whose value can be determined using Rapid Distortion Theory in homogeneous turbulence (Launder et al., 1975; Pope, 2000). $C_{\phi m}$ and $C_T$ are the Kolmogorov and Kolmogorov–Obukhov–Corrsin constants for vertical velocity and air temperature spectra, respectively. For a one-dimensional interpretation of $k$, their common values are $C_{\phi m} = 0.65$ and $C_T = 0.8$ (Sreenivasan, 1995; Sreenivasan and Antonia, 1997; Ishihara et al., 2002; Chung and Matheau, 2012). $\phi_m(\zeta) = [k_{\infty} z / u^*_T] (d\overline{U}/dz)$ is the stability correction function for the mean velocity gradient ($d\overline{U}/dz$), which is only a function of $\zeta$ (Obukhov, 1946; Monin and Obukhov, 1954; Businger and Yaglom, 1971). Using Eq. (6), the first term on the right-hand side of Eq. (5) can now be evaluated as 
\[ \frac{-dT}{dz} \int z^{-1} F_{wT}(k) dk = \frac{3}{4} C_{wT} \epsilon z^0. \] (8) 

### 3.5. Parametrization for the molecular dissipation term

The ratio of the transfer to the molecular dissipation terms in Eq. (5) is:
\[ \frac{\int z^{-1} TR(k') dk'}{\int z^{-1} D_m(k') dk'} = \frac{\int z^{-1} D_i(k') dk'}{\int z^{-1} D_m(k') dk'} = \frac{3}{4} C_{H} C_{m} \kappa_{v} \frac{1}{2} \frac{v_m}{D_m} Pr_{T}^{-1} \phi_{VT}^{-1} Re, \] (13)

where $Re = u_x z / v_m$ is the Reynolds number, $v_m$ is the kinematic molecular viscosity and $\phi_{VT} = \epsilon \kappa_{v} z / u^*_T$ is the normalized dissipation rate for turbulent kinetic energy. For a sufficiently high $Re$, the molecular dissipation term is much smaller than the transfer term and hence can be ignored at $k = 1/z$ (Banerjee and Katul, 2013; Banerjee et al., 2015).

### 3.6. Evaluation of the temperature spectrum

Given these approximations for all three terms on the right-hand side of Eq. (5) and $N_r = -\overline{w^2T^2} dz + VT T$, it can be shown that 
\[ \overline{w^2T^2} dz + VT T = \frac{3}{4} C_{wT} \epsilon z^0 \left( \frac{d\overline{T}}{dz} \right)^2 + \frac{3}{4} Pr_{T}^{-1} C_{H} C_{m} \kappa_{v} \frac{1}{2} \frac{v_m}{D_m} \phi_{VT}^{-1} T_a \] (14)

which can be arranged to yield
\[ \int \frac{k^2 E_{TT}(k') dk'}{k^3} = \frac{3}{4} C_{wT} \epsilon z^0 \left( \frac{d\overline{T}}{dz} \right)^2 + \frac{3}{4} Pr_{T}^{-1} C_{H} C_{m} \kappa_{v} \frac{1}{2} \frac{v_m}{D_m} \phi_{VT}^{-1} T_a \] (15)

where $\phi_{VT}(\zeta) = [k_{\infty} z / T_a] (d\overline{U}/dz)$ is the stability correction function for the mean air temperature gradient, and $T_a = -\overline{w^2T^2}/u^*_T$ again the temperature scaling parameter for the ASL. It can be shown that, by definition, $\phi_{VT}(\zeta)/\phi_{VT}(\zeta) = Pr_{T}$. The $\phi_{VT}$ is a normalized imbalance between production and dissipation rates, defined as:
\[ \phi_{VT} = \frac{\kappa_{v} z}{u_x T_a} VT = \phi_{NL}(\zeta) - \phi_{VT}(\zeta). \] (16)

where $\phi_{NL}(\zeta) = N_r \kappa_{v} z / (u_x T_a)$ is the normalized dissipation rate for half the temperature variance.

Similar to the approach used in Banerjee et al. (2015) for $E_{wT}(k)$, it is assumed that the solution to the air temperature spectrum follows a power law whose form is $E_{TT}(k) = ak^a$ in the range of $[z^{-1}, \infty]$ (Kolmogorov, 1941), that is, $E_{TT}(k) = C_{e} e^{1/k^{2/3}}$, Eq. (10) leads to 
\[ \int_{z^{-1}} \frac{3}{4} C_{H} C_{m} \kappa_{v} \frac{1}{2} \frac{v_m}{D_m} \phi_{VT}^{-1} T_a \] (15)

Here, exponential corrections to $k^{-5/3}$ due to viscous effects are ignored for simplicity given the assumed high Reynolds number (i.e., large separation in scales between $z$ and the Kolmogorov microscale $n$) though these corrections can be included. Also, intermittency corrections to the $k^{-5/3}$ within the inertial subrange scaling are usually small for second-order statistics such as spectra as discussed elsewhere (Sreenivasan and Antonia, 1997) and are ignored here for simplicity, though they can also be accommodated. With these approximations, 
\[ \int_{z^{-1}} \frac{3}{4} C_{H} C_{m} \kappa_{v} \frac{1}{2} \frac{v_m}{D_m} \phi_{VT}^{-1} T_a \] (15)

where, for convenience, $\phi_{b}$ is given as
\[ \phi_{b} = 2 \left( \frac{3}{4} C_{wT} \epsilon z^0 \left( \frac{d\overline{T}}{dz} \right)^2 + \frac{3}{4} Pr_{T}^{-1} C_{H} C_{m} \kappa_{v} \frac{1}{2} \frac{v_m}{D_m} \phi_{VT}^{-1} T_a \right) = \phi_{a1} + \phi_{a2} + \phi_{a3}. \] (18)
That is, $\phi_a$ arises from the superposition of three processes given as

$$
\phi_a = 2 \left( \frac{2 \phi_T \kappa_{T}^{-1} C_{\mu} \phi^{-1}}{3} \right)
$$
\[(19)\]

$$
\phi_{\alpha_2} = 2 \left( \frac{2 \phi_T \kappa_{T}^{-1} C_{\mu} \phi^{-1}}{3} \right)
$$
\[(20)\]

$$
\phi_{\alpha_3} = 2 \left( \frac{2 \phi_T \kappa_{T}^{-1} C_{\mu} \phi^{-1}}{3} \right)
$$
\[(21)\]

If $\phi_a$ is independent from $z$, then polynomial matching in Eq. (17) yields $b = -1$ and a spectral similarity constant $a = \phi_T T_{2}^{-1}$, so that $E_{TT} = \phi_T T_{2}^{-1} k^{-1}$. That is, the temperature spectra must follow a $-1$ scaling in the range of $[0, z^{-1}]$ but only when $\phi_a$ is independent from $z$. With this derivation, the objective of identifying atmospheric stability conditions promoting or suppressing the $-1$ power-law scaling in $E_{TT}(k)$ can be readily addressed by analyzing conditions promoting $\phi_a$ independence in $\phi_a$. Contributions from $\phi_{\alpha_1}, \phi_{\alpha_2}$, and $\phi_{\alpha_3}$ to $\phi_a$ across various $\zeta$ ranges are now considered and discussed using the two aforementioned experiments.

4. Results

Figures 1 and 2 present the averaged $E_{TT}(k)$ for each of the eight stability regimes listed in Table 1 using the lake and grass data. Also shown are the $-1$ scaling (in the low-wavenumber range) in all panels and the $-5/3$ scaling (in the high-wavenumber range) in (a) for illustration. Following Kader and Yaglom (1991) and Katul et al. (1995) who showed that $\phi_a \approx 0.9$ under near-neutral conditions, the black lines in Figures 1 and 2 correspond to $E_{TT}(k) = 0.9 T_{2}^{-1} k^{-1}$. Measured $E_{TT}(k)$ in (a)–(e) (i.e. $-0.1 < \zeta < 0.2$) exhibit a distinct $-1$ scaling at the low $k$ and are in acceptable agreement with $\phi_a = 0.9$. A $-1$ scaling is also observed in $E_{uw}(k)$ over the same stability range (not shown) while the energy spectra of the vertical velocity, $E_{uw}(k)$, saturates to a near-constant value at low $k$ (Li et al., 2015a). This saturation in $E_{uw}(k)$ indicates that the measurements are from the ‘eddy surface layer’ according to Drobinoski et al. (2004). In addition, the transition between the $-1$ and the $-5/3$ scaling occurs in the vicinity of $z_1 = 1$ supporting the evaluation of the spectral budget at $k = z^{-1}$ (Eq. (5)) and assuming inertial subrange scaling applies for $k > 1$ consistent with the results in Kader and Yaglom (1991). However, for (f)–(h), the range of $k$ following a $-1$ scaling is diminished and measured $E_{TT}(k)$ at low $k$ appreciably deviate from $E_{TT}(k) = 0.9 T_{2}^{-1} k^{-1}$. It is also noted here that the scaling in the inertial subrange seems to deviate from its classical $-5/3$ value under near-neutral conditions (e.g. regimes $d$ and $e$), which is consistent with other long-term field measurements (Grachev et al., 2013).

Figure 3 shows variations of $\phi_a$ and its three components, $\phi_{\alpha_1}, \phi_{\alpha_2}$, and $\phi_{\alpha_3}$, when $-0.1 < \zeta < 0.2$ (i.e. in stability regimes where the $-1$ scaling is observed). To calculate $\phi_{\alpha_1}$, $\phi_{\alpha_2}$, and $\phi_{\alpha_3}$, the Businger–Dyer relations for $\phi_{\alpha_0}(\zeta)$ are assumed (Businger et al., 1971; Businger, 1988). The $\phi_T(\zeta)$ is calculated as $\phi_T(\zeta) = P_{\alpha_0}(\zeta) \phi_{\alpha_0}(\zeta) = P_{\alpha_0}(\zeta) Q^{-1} \phi_{\alpha_0}(\zeta)$, where $Q$ is determined from Eq. (7). This aforementioned determination of $Q$ must be viewed as a simplification whose (minor) consequences are explored in Appendix B. In addition, the budget equation for the turbulent kinetic energy is assumed to be in equilibrium (Katul et al., 2013, 2014; Li et al., 2015b) so that $\phi_{B}(\zeta) = \phi_{\alpha_0}(\zeta) - \zeta$ (Banerjee et al., 2015, gives experimental evaluation of this assumption). The $\phi_{B}$ is determined for reference so that $\phi_{\alpha_0}$, the sum of $\phi_{\alpha_1}, \phi_{\alpha_2}$, and $\phi_{\alpha_3}$, is 0.9 consistent with the value in Figures 1 and 2.

It is evident from Figure 3 that $\phi_{\alpha_1}$ and $\phi_{\alpha_2}$ are large but roughly cancel each other. Nonetheless, the sum of $\phi_{\alpha_1}$ and $\phi_{\alpha_2}$ still exhibits a weak dependence on $\zeta$ and hence on $z$. Consequently, if $\phi_{\alpha_3}$ is assumed to be 0 (i.e. $\phi_T = 0$), $\phi_{\alpha_0}$ exhibits a weak dependence on $z$ and the resulting $E_{TT}(k)$ from Eq. (17) cannot strictly follow a $-1$ scaling for $k$ within $[0, z^{-1}]$. Within the limits of the theory here, the only mechanism that counters such $z$ dependence is a non-zero $\phi_{\alpha_3}$. The $\phi_{\alpha_3}$ must slightly increase with increasing instability as shown in Figure 3 to yield a $\phi_a$ independent of $z$ for near-neutral stability conditions. As such,

$$
\phi_{TT} = \phi_{\alpha_0} \left( \frac{2 \phi_T^{-1} \kappa_{T}^{-1} C_{\mu} \phi^{-1}}{3} \right)
$$
\[(22)\]

cannot be 0 but must increase as $\zeta$ decreases (i.e. as the atmosphere becomes unstable).

Although the $-1$ scaling of $E_{TT}(k)$ is diminished or absent altogether in stability regimes $f$ to $h$ (Figures 1 and 2), analysis similar to Figure 3 is conducted over the range of $\zeta$ covered by these unstable regimes simply to illustrate the requirement for $\phi_{\alpha_3}$ (and hence $\phi_{TT}$) if a $-1$ scaling were to hold for these unstable regimes. Figure 4 shows the variations of $\phi_a$ and its three different components, $\phi_{\alpha_1}, \phi_{\alpha_2}$, and $\phi_{\alpha_3}$, when $-5 < \zeta < -0.1$. Again, $\phi_{\alpha_3}$ is calculated so that $\phi_{\alpha_0}$ (the sum of $\phi_{\alpha_1}, \phi_{\alpha_2}$, and $\phi_{\alpha_3}$) equals to 0.9. As can be seen, the sum of $\phi_{\alpha_1}$ and $\phi_{\alpha_3}$ depends on $\zeta$ and hence on $z$, especially when $\zeta < -1$. In addition, the sum of $\phi_{\alpha_1}$ and $\phi_{\alpha_3}$ approaches 0 at small $\zeta$. To achieve a continuous $\phi_a$ that is also independent of $z$, $\phi_{\alpha_3}$ must increase significantly with increasing $-\zeta$ (i.e. as the atmosphere becomes more unstable). As a result, $\phi_{TT}$ must increase concomitantly as $-\zeta$ increases.

Figure 5 shows calculated $\phi_{TT}$ from $\phi_{\alpha_3}$ promoting $-1$ scaling (Eq. (22)), as compared to estimated $\phi_{TT}$ from Eq. (16) with $N_{\alpha}$ inferred from the third-order mixed velocity–temperature structure function $D_{TT}(r)$. Similar results are obtained with $N_{\alpha}$ inferred from second-order temperature structure function (Appendix A). The experiments here suggest an increasing $\phi_{TT}$ as $\zeta$ decreases when $-0.1 < \zeta < 0.2$ (regimes $a$ to $e$), which is because $\phi_{TT}$ is mostly negative under stable conditions but switches to positive under unstable conditions over this stability range. Hence, the experiments here confirm that $\phi_{TT}$ is non-zero and can vary with $\zeta$. A $\phi_{TT}$ switching sign in the near-neutral limit as $\zeta$ crosses from unstable to stable conditions (and conversely) appears necessary for the existence of a $-1$ scaling in $E_{TT}(k)$ even when the ASL flow is idealized. The agreement between the model calculations and experimental results during $0.1 < \zeta < 0.2$ (regime $a$) seems to be satisfactory but poorer than in the other regimes where the $-1$ scaling exists, which is probably also due to the relatively smaller number of segments (particularly the lake data) in this regime.

When $-5 < \zeta < -0.1$ (regimes $f$ to $h$), the conditions promoting the $-1$ scaling in $E_{TT}(k)$ are no longer satisfied. The experiments indicate that the normalized imbalance between production and dissipation rates of half the temperature variance ($\phi_{TT}$) fall short of what is needed to maintain a $-1$ power-law scaling for the stability range of $-5 < \zeta < -0.1$. This is the main reason why a $-1$ power-law scaling is absent for such stability regimes. The $-5 < \zeta < -0.5$ are not presented in Figure 5 since they show an even larger discrepancy between conditions promoting $-1$ scaling in $E_{TT}(k)$ and the experimental data as compared to those when $-0.5 < \zeta < -0.1$.

5. Discussion and conclusions

The stability conditions promoting or suppressing the $-1$ power-law scaling in the low $k$ range of $E_{TT}(k)$ are examined experimentally using ASL data collected above a lake and a grass surface and theoretically using simplifications to an air temperature spectral budget. In agreement with the theoretical predictions, the experiments reveal a $-1$ scaling law for $E_{TT}(k)$ under mildly unstable to mildly stable conditions (when


Figure 1. The measured normalized spectra of air temperature ($E_{TT}(k)$) for the eight stability regimes from the lake data. (a)–(h) correspond to the stability regimes designated as a to h in Table 1. All spectra are averaged over all segments included in each stability regime. The black lines are $E_{TT} = 0.9T^2k^{-1}$ (i.e. $\phi_u = 0.9$). The $E_{TT}(k)$ in (a)–(e) show a distinct $-1$ scaling in the low-wavenumber range and the transitions between $-1$ and $-5/3$ scaling occur in the vicinity of $kz = 1$.

$-0.1 < \zeta < 0.2$). As instability increases (when $-5 < \zeta < -0.1$), the measured $-1$ scaling in $E_{TT}(k)$ diminishes or censors consistent with theoretical expectations. The theory underscores the significant role of $\phi_{VT}$, the normalized imbalance between the production and dissipation rates of half the temperature variance, in controlling the $-1$ scaling. When $-0.1 < \zeta < 0.2$, $\phi_{VT}$ estimated from experiments shows a weak dependence on $z$, which results in a $z$-independent $\phi_u$ and a $-1$ scaling in the low $k$ range of $E_{TT}(k)$. When $-5 < \zeta < -0.1$, $\phi_{VT}$ estimated from experiments appears insensitive to $z/L$, thereby suppressing the onset of a $-1$ scaling in $E_{TT}(k)$.

The results presented here have two implications. First, the fact that $\phi_{VT}$ plays a role in the $-1$ scaling of $E_{TT}(k)$ implies that possible connections between $E_{uu}(k)$ and $E_{TT}(k)$ are more complicated than what prior dimensional analysis may suggest. Banerjee and Katul (2013) and Banerjee et al. (2015) examined the $-1$ scaling in $E_{uu}(k)$ using a similar approach but did not retain a finite turbulent transport term in their budget equation for turbulent kinetic energy, suggesting an insignificant role of the imbalance between the production and dissipation rates of turbulent kinetic energy in controlling the existence of $-1$ scaling of $E_{uu}(k)$. As a result, the existence of $-1$ scaling in $E_{uu}(k)$ does not automatically imply a $-1$ scaling in $E_{TT}(k)$ at low $k$. This might explain why some experiments (including the lake/grassland experiments presented here) reported concurrent $-1$ scaling in $E_{uu}(k)$ and $E_{TT}(k)$ but not other experiments (Pond et al., 1966).

The spectral/co-spectral budgets proposed here, by virtue of their formulation, explain why a $-1$ scaling in $E_{TT}(k)$ must extend to $kz \sim 1$ whereas previous dimensional analysis assumes the $-1$ scaling holds for $kz << 1$ (Kader and Yaglom, 1991).

Second, given that

\[ \int_0^\infty E_{TT}(k) \, dk' = \overline{T^2} = \sigma^2_T, \]
$k^{-1}$ Scaling of Temperature Spectra in the ASL

$E_T / \sigma_T^2$

Figure 2. As Figure 1 but for the grass data.

Figure 3. The variations of $\phi_1$, $\phi_2$, and $\phi_3$ with $\zeta$. The dashed line corresponds to a value of 0.9. The range of $\zeta$ shown here includes regimes a to e in Table 1.

the $-1$ scaling in the $E_{TT}(k)$ will result in a dependence of $\sigma_T$ on the scale that characterizes the largest turbulent disturbances ($kz << 1$) such as the atmospheric boundary-layer height. A similar result but for $\sigma_u$ has been presented elsewhere (Banerjee and Katul, 2013; Banerjee et al., 2015). A dependence of $\sigma_T$ on the atmospheric boundary-layer height might explain the large scatter of $\sigma_T / T^*$ under near-neutral conditions, especially when the sensible heat flux or $T^*$ is finite (i.e. not too small). Future studies involving comparisons between $E_{TT}(k)$ and the spectra of other scalars such as $q$ (i.e. water vapour) and CO$_2$ might shed light on how the $-1$ scaling in the $E_{TT}(k)$ affects scalar similarity and flux-variance relations.

The $-1$ scaling examined here is different from the classical $-1$ scaling in the 'viscous–convective' subrange at large Schmidt or Prandtl numbers (Davidson et al., 2012). The $-1$ scaling in $E_{TT}$ here describes the low-wavenumber range (from $k = 0$ to $k = 1/z$, where $z$ is the height above the ground) and is not connected to a large molecular Prandtl number (in fact the Prandtl number of the air is only about 0.71); while the $-1$ scaling in the viscous–convective subrange spans from $k\eta = 0.2$ to $k\eta = 1$ (Bos and Bertoglio, 2013) and requires a molecular
Heusinkveld for providing the grass data collected at Wageningen University.

Appendix A

Computation of dissipation rates from experimental data

In this study, two methods are used to calculate the dissipation rate of half the temperature variance $N_e$. The first method calculates $N_e$ from the second-order temperature structure function following $N_e = -(0.3125)[D_{TT}(r)]r^{3/2}/[e^{1/3}a]$ and the second method calculates $N_e$ from the third-order mixed velocity–temperature structure function $D_{TTr}(r)$ following $N_e = -(3/4)[D_{TTr}(r)]/r$ (Stull, 1988) for each run, where $r$ is the separation distance aligned along the mean wind direction and inferred from Taylor’s frozen turbulence hypothesis (Taylor, 1938),

$$D_{TT}(r) = \frac{\langle T'(x+r) - T(x) \rangle^2}{\langle T'(x) \rangle^2}$$

and
d$$D_{TTr}(r) = \frac{\langle T'(x+r) - T(x) \rangle^2}{\langle u'(x+r) - u(x) \rangle^2}$$.

Estimation of $N_e$ from the second-order structure function requires the dissipation rate of turbulent kinetic energy $\epsilon$, which can be also computed from either second-order or third-order longitudinal velocity structure functions (Hsieh and Katul, 1997). Figure A1 shows comparisons of normalized $N_e$ ($\phi N_e$) and $\epsilon$ ($\phi \epsilon$) estimated from second-order and third-order structure functions. Despite some scatter, especially for the grass data, it is clear that the two estimates generally agree with each other, consistent with other findings (Bou-Zeid et al., 2010). When $N_e$ and $\epsilon$ estimated from second-order structure functions are used, conditions promoting the $k^{-1}$ scaling are still satisfied when $-0.1 < \xi < 0.2$ (not shown here but can be inferred from Figures 5 and A1).

The scatter seen in the grass data in Figure A1 is probably due to a combination of limited inertial subrange and high turbulence intensity. For the grass data, $r = z/5$ is chosen, as compared to $r = z/2$ for the lake data. This is because $D_{TT}$ and $D_{uu}$ clearly do not follow the $r^{-2/3}$ scaling at $r = z/2$ (not shown) and hence a smaller $r$ has to be used to be in the inertial subrange. Given that the averaging path of a sonic anemometer is about 15 cm, the inertial subrange is hence relatively narrow for the grass data, which makes it more challenging to estimate dissipation rates from structure functions. In addition, the high turbulence intensities also modulate the relationship between dissipation rates from structure functions (e.g. the coefficient 0.3125 in the calculation of $N_e$ from $D_{TT}$ depends on the turbulent intensity, as shown in Hsieh and Katul, 1997). In this study, the turbulence intensity is restricted to be smaller than 0.5 as recommended by Stull (1988) to justify the use of Taylor’s frozen turbulence hypothesis. When a more stringent criterion is used (e.g. the turbulence intensity needs to be smaller than 0.25), the scatter is significantly reduced (not shown here). It is stressed that, despite the scatter, the experimental data agree with model calculations fairly well, as can be seen from Figure 5.

Appendix B

Computation of $Q$ in the co-spectral budget model

In the co-spectral budget model used to estimate the production term (Eq. (6)), it is assumed that the budget equation for half the temperature variance is in equilibrium or $VT = 0$ so as to obtain $Q$. The $Q$ is needed for computing $C_T$ and $P_T$. To assess the effects of a non-zero VT on $Q$, a revised (and lengthy) expression for $Q$ is derived by retaining VT. This expression is given as

$$Q = \left[1 - \frac{1}{1 - C_T} \frac{C_T}{C_0} (\phi_m(\xi) - \xi) \left(1 + \frac{\phi VT}{\phi T}\right) \right].$$

**Figure 5.** The normalized imbalance between production and dissipation rate of half the temperature variance ($\phi \epsilon$). The line is calculated using Eq. (22) with $\phi N_e$ shown in Figures 3 and 4. The circles (squares) are estimated from the lake (grassland) data using Eq. (16) with $N_e$ inferred from the third-order mixed velocity–temperature structure function $D_{TTr}(r)$.

**Figure 4.** As Figure 3, but for regimes $f$ to $h$ in Table 1.
It is clear from Eq. (B1) that the new \( Q \) further depends on the ratio of \( \phi_{VT}/\phi_T \), suggesting that a non-zero \( \phi_{VT} \) alters the atmospheric stability impact on \( P_T \) and \( C_{wT} \) from what was previously assumed. As a result, \( \phi_{a1}, \phi_{a2}, \) and \( \phi_{a3} \) are now all dependent on \( \phi_{VT} \) and solving \( \phi_{a1} + \phi_{a2} + \phi_{a3} = 0.9 \) to obtain \( \phi_{VT} \) requires an iterative procedure. Figure B1 shows the calculation of \( \phi_{VT} \) using multiple iterations compared to the lake data when \( -0.5 < \zeta < 0.2 \) (i.e. similar to Figure 5). As can be seen, the iterations converge very quickly since the results with five iterations are almost identical to those with 50 iterations (the two lines overlap on Figure B1). The revised calculations are not significantly impacted and remain consistent with measurements over the stability range \( -0.1 < \zeta < 0.2 \) where the \(-1\) scaling is observed.

References


