Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 66, No. 2, November 15, 1978

Competition, Decision, and Consensus

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1. INTRODUCTION

The following problem, in one form or another, has intrigued philosophers and scientists for hundreds of years: How do arbitrarily many individuals, populations, or states, each obeying unique and personal laws, ever succeed in harmoniously interacting with each other to form some sort of stable society, or collective mode of behavior? Otherwise expressed, if each individual obeys complex laws, and is ignorant of other individuals except via locally received signals, how is social chaos averted? How can local ignorance and global order, or consensus, be reconciled? This paper considers a class of systems in which this dilemma is overcome.

We begin by asking what design constraints must be imposed on a system of competing populations in order that it be able to generate a global limiting pattern, or decision, in response to arbitrary initial data? This paper proves that global pattern formation occurs in systems of the form

$$\dot{x}_i = a_i(x)\,[b_i(x_i) - c(x)] \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $i = 1, 2, \ldots, n$. Such systems can have any number of competing populations ($n \geq 2$), any interpopulation signal functions $b_i(x_i)$, any mean competition function, or adaptation level, $c(x)$, and any state-dependent amplifications $a_i(x)$ of the competitive balance. Systems of type (1), which can be highly nonlinear, arise in many biological applications, such as pattern formation in development [1, 2], the transformation and short-term storage of sensory data in psychophysiology [3–6], competitive interactions among groups or communities is ecology and sociology [1, 7], decision-making in a parallel processor [1, 3, 4], and related areas. Recently considerable interest has been focused on the question: How simple can a system be and still generate "chaotic" behavior? This question is motivated both by a desire to understand turbulence in fluids and by a desire to understand how organized biological interactions can break down under parametric changes [8, 9]. This paper considers the converse

* Supported in part by the National Science Foundation (NSF MCS 77-02958).

0022-247X/78/0662–0470$02.00/0

question: How complicated can a system be and still generate order? The results herein hold because, despite essentially arbitrary irregularities and nonlinearities in local system design, there exists a powerful symmetry in the global rules that bind together the interacting populations. This symmetry is expressed by the existence of a state-dependent mean competition function, or adaptation level, $c(x)$. It can be caused by the existence of long-range interpopulation interactions that have comparable effects on all populations, but otherwise represent an essentially arbitrary competition. The results herein therefore suggest that a breakdown of symmetry in competitive systems, say due to the existence of asymmetric biases in short-range interpopulation interactions, is a basic cause of oscillations and chaos in these systems; cf. [10, 11], where this fact is illustrated by the voting paradox in Volterra–Lotka systems. There appears to exist a complementary, or trade-off, between how global the adaptation level ("communal understanding") is and how freely local signals ("individual differences") can be chosen without destroying global consensus.

The main result is proved by explicating as a mathematical method a main theme about competitive systems; namely, who is winning the competition? The method keeps track of which population is being maximally enhanced as time goes on. When a different population starts to be maximally enhanced, the system "decides" to enhance the new population, or "jumps" between populations. These jumps are a source of system oscillations. Were the jumps never to cease, approximately periodic or even chaotic behavior could ensue. The theorem guarantees, however, that after a time interval of perhaps very complicated, and even seemingly random oscillations, the decision process is essentially completed, and the system approaches the final pattern in an orderly fashion, even if the jumps do not cease. Reference [12] applies this method to a less general problem and reviews earlier work in this direction.

By studying system "jumps" or "decisions", three themes of general interest emerge. First, one analyses the continuous nonlinear system by studying the discrete series of jumps that it induces. Second, although the continuous system describes parallel interactions, it can be analysed in terms of its serial jumps. Third, the analysis of jumps shows that there exists a sequence of nested "dynamic boundaries" that appear as the system evolves. By this is meant the following. Suppose that $x_i(t) \in [0, B]$ for all $t \geqslant 0$. There exists a sequence of nested partitions $E_{j1}^{(i)} \oplus E_{j2}^{(i)} \oplus \cdots \oplus E_{jm_j}^{(i)}$ of $[0, B]$ into half-open intervals $E_{jk}^{(i)}$, $j = 1, 2, \ldots$, such that after time T_1, $x_i(t)$ remains in some interval $E_{1k_1}^{(i)}$, after time T_2, $x_i(t)$ remains in some interval $E_{2k_2}^{(i)} \subset E_{1k_1}^{(i)}$, and so on. The endpoints of each interval define a "dynamic boundary" beyond which $x_i(t)$ cannot migrate. As the jumps continue, the system "decides" to restrict $x_i(t)$ to ever finite intervals, until as $t \to \infty$ a definite limiting value for $x_i(\infty)$ is established. The existence of these dynamic boundaries is a purely nonlinear effect that arises from the interaction of a nonlinear signal function and a nonlinear mass action law within a competitive geometry.

2. Competitive Systems

The simplest competitive feedback interaction among n populations v_i with activities $x_i(t)$ that obey mass action dynamics is

$$\dot{x}_i = -Ax_i + (B - x_i)[f(x_i) + I_i] - x_i\left[\sum_{k \neq i} f(x_k) + J_i\right], \tag{2}$$

$i = 1, 2, \ldots, n$. System (1) has the following interpretation. Let each population v_i have B excitable sites, of which $x_i(t)$ are excited and $B - x_i(t)$ are unexcited at time t. Let a signal $f(x_i(t))$ be generated by the excited sites of v_i. Then term $-Ax_i$ describes the spontaneous switching-off of excitation at rate A; term $(B - x_i) f(x_i)$ describes the switching on of unexcited sites by a positive feedback signal from v_i to itself; term $-x_i f(x_k)$ describes the switching off of excited sites at v_i by a competitive (or negative) feedback signal from v_k to v_i, $k \neq i$; and terms $(B - x_i) I_i$ and $-x_i J_i$ describe the effects of excitatory input I_i and inhibitory input J_i to v_i. This system was first analysed in [3] in a psychophysiological content. In neural terminology, (2) describes the simplest recurrent (feedback) on-center (excite v_i) off-surround (inhibit all v_k, $k \neq i$) interaction of shunting, or passive membrane (or mass action) dynamics and was used to understand aspects of how input patterns to fields of neocortical feature detectors are transformed before they are stored in short-term memory. The results classify ways the choice of the signal function $f(w)$ influences this transformation. The problem studied was as follows: Suppose that the inputs $(I_1, I_2, \ldots, I_n)$ and $(J_1, J_2, \ldots, J_n)$ act before time $t = 0$ to establish an initial pattern of activity $x = (x_1, x_2, \ldots, x_n)$ at $t = 0$. If these inputs are switched off at time $t = 0$, how does the network

$$\dot{x}_i = -Ax_i + (B - x_i) f(x_i) - x_i \sum_{k \neq i} f(x_k) \tag{3}$$

determine the behavior of $x(t)$ as $t \to \infty$? In particular, do there exist choices of $f(w)$ such that system (3) stores biologically important patterns, yet prevents noise amplification via its positive feedback loops?

This latter problem arose because systems such as (2) solve an ubiquitous biological problem: the noise-saturation dilemma. This dilemma asks how a system of noisy populations with finitely many excitable sites can process continuously fluctuating input patterns? When the input patterns are small, they can get lost in the noise. When the inputs are large, they can saturate the system by exciting all of its excitable sites. Competitive systems such as (2) elegantly solve this problem, by balancing between the two extremes of noise and saturation. The choice of $f(w)$ helps to establish this balance; in particular, sigmoid or S-shaped signal functions $f(w)$ balance between too little vs too much noise suppression. When the competitive balance breaks down, either too much or too little noise suppression can occur, thereby leading to various pathologies, such as "seizures" [3, 4, 11].

Not all competitive systems are as simple as (2). A problem of classification is hereby suggested: How do competitive systems that differ in terms of their mass action dynamics, competitive geometry, and statistics of interpopulation signaling generate different transformations of their initial data while trying to overcome the noise-saturation dilemma in their own way? Papers [1] and [12] discuss this classification problem and review some of the transformations that have already been studied.

Systems (1) are a significant generalization of (3) and of the systems studied in [12]. For example, (1) includes systems of the form

$$\dot{x}_i = -A_i x_i + (B_i - x_i)[f_i(x_i) + I_i] - x_i\left[\sum_{k \neq i} f_k(x_k) + J_i\right], \tag{4}$$

in which each population v_i can have different decay rates A_i, different numbers of excitable sites B_i, different signal functions $f_i(x_i)$, and different constant (or tonic) inputs I_i and J_i. System (4) becomes (1) given

$$a_i(x) = x_i, \tag{5}$$

$$b_i(x_i) = x_i^{-1}[B_i f_i(x_i) + I_i] - A_i - I_i - J_i, \tag{6}$$

and

$$c(x) = \sum_{k=1}^{n} f_k(x_k). \tag{7}$$

System (1) also includes generalized Volterra–Lotka systems

$$\dot{x}_i = D_i(x)\left[1 - \sum_{k=1}^{n} f_k(x_k) E_{ki}(x)\right], \tag{8}$$

given state-dependent competition coefficients of the form $E_{ki}(x) = F_k(x_k) G_i(x_i)$ [7, 10]. Such competition coefficients describe statistically independent couplings between populations v_k and v_i via the statistically independent factors $F_k(x_k)$ and $G_i(x_i)$. An alternative description of this system is that the vector function

$$G(x) = (G_1(x_1), G_2(x_2), \ldots, G_n(x_n))$$

describes a state-dependent preference order among the populations. System (8) reduces to (1) given the identifications

$$a_i(x) = D_i(x) G_i(x_i), \tag{9}$$

$$b_i(x_i) = G_i^{-1}(x_i), \tag{10}$$

and

$$c(x) = \sum_{k=1}^{n} f_k(x_k) F_k(x_k). \tag{11}$$

The theorem also holds for such complex nonlinear examples as

$$a_i(x) = x_i^{A_i} \exp\left(\sum_{k=1}^{n} x_k^{B_{ki}}\right), \tag{12}$$

$$b_i(x_i) = \sin(C_i x_i^{D_i} - E_i), \tag{13}$$

and

$$c(x) = \sum_{k=1}^{n} \exp(G_k x_k^{H_k}), \tag{14}$$

where all the coefficients $A_i, \ldots, H_i$ are positive. Indeed, the theorem holds for essentially any physically meaningful choice of the functions $a_i(x)$, $b_i(x_i)$, and $c(x)$, and thereby describes a robust design that guarantees global pattern formation by competitive systems.

3. Global Consensus Theorem

Below are considered systems of the form

$$\dot{x}_i = a_i(x)[b_i(x_i) - c(x)], \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, and n is any integer greater than 1. To state the main theorem, the following hypotheses will be needed:

(I) *Smoothness:*

(a) $a_i(x)$ is continuous for $x \geqslant 0$;

(b) $b_i(x_i)$ is either continuous with piecewise derivatives for $x_i \geqslant 0$, or is continuous with piecewise derivatives for $x_i > 0$ and $b_i(0) = \infty$;

(c) $c(x)$ is continuous with piecewise derivatives for $x \geqslant 0$.

(II) *Nonnegativity:*

$$a_i(x) > 0 \quad \text{if} \quad x_i > 0 \quad \text{and} \quad x_j \geqslant 0, \quad j \neq i, \tag{15a}$$

$$a_i(x) = 0 \quad \text{if} \quad x_i = 0 \quad \text{and} \quad x_j \geqslant 0, \quad j \neq i. \tag{15b}$$

Moreover, there exists a function $\bar{a}_i(x_i)$ such that, for sufficiently small $\lambda > 0$, $\bar{a}_i(x_i) \geqslant a_i(x)$ if $x \in [0, \lambda]^n$ and

$$\int_0^\lambda \frac{dw}{\bar{a}_i(w)} = \infty. \tag{16}$$

(III) *Boundedness:*

$$\limsup_{w\to\infty} b_i(w) < c(0, 0,..., \infty,..., 0, 0) \tag{17}$$

where "∞" occurs in the ith entry, $i = 1, 2,..., n$.

(IV) *Competition:*

$$\frac{\partial c}{\partial x_k} \geqslant 0, \qquad k = 1, 2,..., n. \tag{18}$$

Given essentially any functions that satisfy (15)–(18), we prove that any initial data $x(0) \geqslant 0$ generates an asymptotic pattern, or decision, $x(\infty)$ such that $0 \leqslant x(\infty) < \infty$. In general, there can exist nondenumerably many limit values that $x(\infty)$ might assume, but the analysis of jumps provides considerable information about the dependence of $x(\infty)$ on $x(0)$. There exists a highly degenerate and unlikely situation, however, in which the possibility of oscillations as $t \to \infty$ has bot been ruled out. Even in this rare case, however, all the signals $b_i(x_i(t))$ have limits as $t \to \infty$. These signals are the only observable data that the states about one another, so that global consensus of observables is always reached. Moreover, even if oscillations in certain $x_i(t)$ persist, they become arbitrarily slow as $t \to \infty$, so that for all practical purposes (e.g., measurements taken over one "generation" at large values of t), limits are always achieved. Whether these slow oscillations ever do occur remains an open problem. To state the theorem in its present form, three further concepts will be introduced.

DEFINITION 1. System (1) is said to obey the *oscillation condition* if there exists a constant b^* and three signal functions, labelled $b_1(w)$, $b_2(w)$, and $b_3(w)$ for definiteness, such that

(V) $b_1(w) = b^*$ for all $w \in W_1$, where W_1 is an interval of positive length within the range of x_1;

(VI) there exist increasing infinite sequences $\{p_{2i}\}$ and $\{v_{2i}\}$ converging at w_2^*, and all in the range of x_2, such that each p_{2i} is a local maximum of b_2, each v_{2i} is a local minimum of b_2, each $b_2(p_{2i}) > b^*$; and $\lim_{k\to\infty} b_2(p_{2k}) = \lim_{k\to\infty} b_2(v_{2k}) = b^*$; and

(VII) there exists a decreasing infinite sequence $\{q_{3i}\}$ converging at w_3^*, and all in the range of x_3, such that $b_3(q_{3i}) < b^*$ for every $i = 1, 2,...$ and $\lim_{k\to\alpha} b_3(q_{3i}) = b^*$.

DEFINITION 2. System (1) achieves *weak global consensus* (or *weak global pattern formation*) if, given any $x(0) \geqslant 0$, all the limits $b_i(x_i(\infty)) = \lim_{t\to\infty} b_i(x_i(t))$ exist, $i = 1, 2,..., n$.

DEFINITION 3. System (1) achieves *strong global consensus* (or *strong global pattern formation*) if, given any $x(0) \geqslant 0$, all the limits $x_i(\infty) = \lim_{j\to\infty} x_i(t)$ exist, $i = 1, 2,..., n$.

THEOREM 1 (Global Consensus). *Any system of form* (1) *whose functions satisfy properties* (I)–(IV) *achieves weak global consensus. Moreover, since each* $b_i(x_i(\infty)) = c(x(\infty))$, *any oscillations that might occur become arbitrarily slow as* $t \to \infty$. *Any system of form* (1) *whose functions satisfy properties* (I)–(IV), *and do not satisfy the oscillation conditions* (V)–(VII), *achieves strong global consensus.*

Remarks. Since the oscillation condition requires at least three signals, any 2-dimensional system of type (1) achieves strong global consensus. Moreover, since the oscillation condition requires b_2 to oscillate infinitely often in a compact interval, and b_2^* to identically equal $b_1(w)$ for all $w \in W_1$, essentially any biologically interesting system of type (1) achieves strong global consensus. For example, any system whose signals are built up from arbitrary finite numbers of random factors within each population achieves strong global consensus; cf. [12, Section 2]. Strong global consensus is a generic property. The main facts are summarized by the following corollary.

COROLLARY 1. *Any system of type* (1) *which satisfies properties* (I)–(IV), *and whose signal functions* b_i *possess finitely many local maxima, or intervals of local maxima, within the range of* x_i, *achieves strong global consensus. In particular, if the signals are real-analytic functions, then strong global consensus is achieved.*

The following corollaries are found when Theorem 1 is applied to competitive mass-action networks and to Volterra–Lotka systems.

COROLLARY 2. *Let system* (4) *possess signal functions* $f_i(x_i)$ *that are continuous, monotone nondecreasing, and have piecewise derivatives for* $x_i \in [0, B_i]$, $i = 1, 2,..., n$. *Then weak global consensus is achieved. If moreover,* $x_i^{-1} f_i(x_i)$ *has finitely many local maxima, or intervals of local maxima for* $x_i \in [0, B_i]$, $i = 1, 2,..., n$, *then strong global consensus is achieved.*

Remark. Corollary 2 generalizes the limit theorems in [4].

COROLLARY 3. *Let system* (8) *with* $E_{ki}(x) = F_k(x_k)\, G_i(x_i)$ *have a continuous* $D_i(x)$ *which is positive unless* $x_i = 0$; *continuous functions* $G_i(x_i)$ *that are positive except possibly at* $x_i = 0$, *and which possess piecewise derivatives; continuous functions* $f_k(x_k)$ *and* $F_k(x_k)$ *such that* $f_k(x_k)\, F_k(x_k)$ *is monotone nonincreasing with piecewise derivatives; and let* (15)–(17) *hold with the identifications* (9)–(11). *Then weak global consensus is achieved. If, moreover* $G_i(x_i)$ *has finitely many local minima, or intervals of local minima, within the range of* $x_i(t)$, *then strong global consensus is achieved.*

Proof of Theorem. The theorem will first be proved for the case that all $b_i \equiv b$. This proof can then be adapted to the case of arbitrary b_i. First one notes by (15) and (16) that if $x_i(0) > 0$ then $x_i(t) > 0$ for $t \geqslant 0$ [7]. If $x_i(0) = 0$, population v_i can be deleted from the network without loss of generality. Hence we restrict attention below to the case of positive initial data. The proof consists of three stages: I. Ignition, II. Jump Sequence (or Iterated Local Decisions), and III. Coda (or Global Consensus).

I. Define the functions

$$M_i(t) = b(x_i(t)) - c(x(t)) \tag{19}$$

and

$$M(t) = \max\{M_k(t): k = 1, 2,..., n\}. \tag{20}$$

"Ignition" means that either $M(t) \leqslant 0$ for all $t \geqslant 0$, or that there exists a $t = T$ such that

$$M(T) \geqslant 0 \quad \text{implies} \quad M(t) \geqslant 0 \quad \text{for } t \geqslant T. \tag{21}$$

To prove (21) it suffices to show that if at any time $t = S$, $M(S) = 0$, then $\dot{M}(S) \geqslant 0$. By (19), if $M(S) = M_i(S)$, then

$$\dot{M}(S) = b'(x_i(S))\, \dot{x}_i(S) - \sum_{k=1}^{n} \frac{\partial c}{\partial x_k}(x(S))\, \dot{x}_k(S).$$

Since $\dot{x}_i(S) = 0 \geqslant \dot{x}_k(S)$, $k = 1, 2,..., n$, (1) and (18) imply that $\dot{M}(S) \geqslant 0$.

By the ignition property, either all $\dot{x}_i \leqslant 0$ for $t \geqslant 0$, or there exists a time $t = T$ after which some x_i, perhaps a different one at different times, is always increasing. In the former case, all $x_i(\infty)$ exist, since all x_i are monotone decreasing and, by (16), all x_i are bounded below by 0. It remains only to consider the latter case. Below we therefore assume that $M(0) \geqslant 0$ without loss of generality.

II. By (16) and (17), there exists a $B > 0$ such that $x_i(t) \in [0, B]$ for all $i = 1, 2,..., n$ and $t \geqslant 0$. Consider the graph of $h(w)$ in the interval $[0, B]$. Decompose the graph into *ascending slopes* A_i and *descending slopes* D_i as follows. Consider successively larger w values, $w \geqslant 0$, until for some $w = W$, $b'(W) \neq 0$. Suppose for definiteness that $b'(W) > 0$. Then the ascending slope A_1 is the maximal connected set of w values, including $w = 0$, wherein $b'(w) \geqslant 0$. The descending slope D_1 is the maximal connected set in $[0, B]\backslash - A_1$ that is contiguous to A_1 wherein $b'(w) \leqslant 0$. The ascending slope A_2 is the maximal connected set in $[0, B] - (A_1 \cup D_1)$ that is contiguous to D_1 wherein $b'(w) \geqslant 0$. And so on. Also define $H_j = A_j \cup D_j$, to be the jth *hill* in the graph of $b(w)$. Let $p_j = \max\{w: w \in A_j\}$ be the *peak* of H_j, and $v_j = \max\{w: w \in D_j\}$ be the *valley* of H_j. Also let $P_j = b(p_j)$ be the *height* and $V_j = b(v_j)$ be the *depth* of H_j. Speaking intuitively, $b(x_i(t))$ is the height of x_i at time t, and P_j is the height of the jth hill peak.

A *jump* is said to occur from i to j at time $t = T$ if there exist times S and U such that $M(t) = M_i(t)$ for $S \leqslant t < T$ and $M(t) = M_j(t)$ for $T \leqslant t < U$. The set of *jump variables* $J = \{i: M(t) = M_i(t)$ for some $t \geqslant 0\}$. The set of *persistent jump variables* $J^\infty = \{i: M(t) = M_i(t)$ for some $t = t_{ik}$, $k = 1, 2,...,$ where $\lim_{k\to\infty} t_{ik} = \infty\}$. The set of *j-persistent jump variables* $J_j^\infty = \{i: M(t) = M_i(t)$ and $x_i(t) \in H_j$ for some $t = t_{ik}$, $k = 1, 2,...,$ where $\lim_{k\to\infty} t_{ik} = \infty\}$. When we say that a jump occurs from an $i \in J_j^\infty$ at time $t = T$, we imply that $x_i(T) \in H_j$ at time T. Otherwise expressed, let $I(t)$ be the index i such that $M_i(t) = M(t)$; that is, $M(t) = M_{I(t)}(t)$ for $t \geqslant 0$. Also define $y(t) = x_{I(t)}(t)$ for $t \geqslant 0$. To say that a jump occurs from an $i \in J_j^\infty$ at time $t = T$ means that $y(t) \in H_j$ just before $I(t)$ changes value at time $t = T$.

To test when a jump will occur from i to j at time $t = T$, suppose that $M_i(T) = M_j(T) = M(T)$. A jump will occur from i to j if $\dot{M}_j(T) > \dot{M}_i(T)$. Since

$$\dot{M}_j(T) = b'(x_j(T))\, a_j(x(T))\, M(T) - \dot{c}(T) \tag{22}$$

and

$$\dot{M}_i(T) = b'(x_i(T))\, a_i(x(T))\, M(T) - \dot{c}(T), \tag{23}$$

where $M(T) \geqslant 0$, a jump occurs from i to j if

$$b'(x_j(T))\, a_j(x(T)) > b'(x_i(T))\, a_i(x(T)). \tag{24}$$

Since $a_i(x(T))$ and $a_j(x(T))$ are nonnegative, a jump can never occur from an ascending slope to a descending slope.

Case 1. *Finitely Many Jumps.* If only finitely many jumps occur, then after a finite amount of time goes by, there exists some i, say $i = 1$, such that thereafter $M_1 = M \geqslant 0$. By (1), (15), and (16), $\dot{x}_1 \geqslant 0$, so that x_1 is monotone increasing. By (17), x_1 is also bounded above. Hence the limit $x_1(\infty)$ exists. This limit lies on some ascending or descending slope. Suppose that it lies on an ascending slope, say A_K for definiteness. We will now prove that $\lim_{t\to\infty} c(x(t))$ exists and equals $b^* = b(x_1(\infty))$.

Suppose not. Whenever $\dot{x}_1 \geqslant 0$, $b(x_1) \geqslant c(x)$. Thus if $c(x(t))$ does not converge to b^*, there exists a sequence of increasing times t_k with $\lim_{k\to\infty} t_k = \infty$ such that

$$b^* - c(x(t_k)) \geqslant 2\epsilon, \qquad k = 1, 2,.... \tag{25}$$

Now note that $c(x(t))$ is uniformly continuous for $t \geqslant 0$. This is true because $x(t)$ remains in a compact set R; $c(x)$ is continuous, and hence uniformly continuous for $x \in R$; and, by (1), there exists a constant M, $0 < M < \infty$, such that $|\dot{x}_i(t)| \leqslant M$ for all $t \geqslant 0$ and $i = 1, 2,..., n$. By the uniform continuity of $c(x)$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $x, y \in R$ and $|x - y| < \delta$ implies $|c(x) - c(y)| < \epsilon$. By the mean-value theorem, $|x(t) - x(s)| \leqslant M|t - s|$. Consequently for $|t - s| \leqslant \delta M^{-1}$, $|c(x(t)) - c(x(s))| < \epsilon$, which proves the assertion. Thus by (25) there exists a $\beta > 0$ and a sequence of nonoverlapping intervals $[Y_k, Z_k]$ such that $Z_k - Y_k \geqslant \beta$ for $k = 1, 2, \ldots$, and

$$b^* - c(x(t)) \geqslant \epsilon, \qquad t \in \bigcup_{k=1}^{\infty} [Y_k, Z_k]. \tag{26}$$

Furthermore, since x_1 is monotone increasing, (15) along with the continuity of $a_1(x)$ for $x \in [0, B]^n$, shows that there exists a $\gamma > 0$ such that

$$a_1(x) \geqslant \gamma \tag{27}$$

when x_1 is close to $x_1(\infty)$. By (1), (26), and (27)

$$\dot{x}_1(t) \geqslant \epsilon\gamma \qquad \text{for } t \in \bigcup_{k=1}^{\infty} [Y_k, Z_k]. \tag{28}$$

Since x_1 is monotone increasing, (28) implies that $x_1(\infty) = \infty$, which is impossible. This contradiction proves that

$$\lim_{t\to\infty} c(x(t)) = b^*. \tag{29}$$

Lemma 1 below completes the proof.

Case 2. Infinitely Many Jumps. If jumps do not cease after a finite amount of time, delete from consideration all hills H_j whose J_j^∞ sets are empty; that is, let the process continue until no jumps ever again occur from hills with empty J_j^∞ sets. Relabel the time scale so that $t = 0$ after all such jumps have occurred. Relabel the hills H_j with nonempty J_j^∞ sets so that $j \leqslant k$ iff $P_j \geqslant P_k$.

The idea of the argument below is to show that given any hill, after a sufficient amount of time goes by, all variables get trapped either to the right or to the left of its peak. Since this is true for any hill, eventually each variable gets trapped in the "bowl" between a contiguous descending slope and ascending slope; in particular, eventually no jump variable can cross over a peak from an ascending slope to a descending slope. In general, the x_i do not get trapped in their bowls all at once. First they get trapped in some interval $E_{1k_1}^{(i)} = [u_{1k_1}^{(i)}, v_{1k_1}^{(i)})$ whose boundary values have the largest peak heights $P_1 = b(u_{1k_1}^{(i)}) = b(v_{1k_1}^{(i)})$; later they get trapped in some interval $E_{2k_2}^{(i)} = [u_{2k_2}^{(i)}, v_{2k_2}^{(i)})$ whose boundary values have the largest or the next-to-largest peak heights $b(u_{2k_2}^{(i)})$ and $b(v_{2k_2}^{(i)})$; and so on. These intervals are the nested dynamic boundaries that were discussed in Section 1. The "bowls" are the final set of dynamic boundaries that are established. After the variables get trapped in their bowls, the decision process is essentially complete. Thereafter, $(d/dt)b(y(t))$ changes sign at most once, so that at all large times $b(y(t))$ is monotonic. This Lyapunov-like behavior is then used to complete the proof.

The above heuristic description tacitly assumes that there exists a finite series of peak heights in the graph of $h(w)$. In any "physical" example, this will be true. In general, however, certain peak heights can be limit points of other peak heights. The proof is extended to such cases by using the fact that these "infinitely wiggly hills are arbitrarily small."

First we will consider the physically important case. Suppose that the successive peak heights $P_1 > P_2 > P_3 > \cdots > P_L$ form a discrete series such that only finitely many hills attain any given height. Below we will assume for definiteness that one hill H_i has height P_i, but the argument immediately generalizes to the case in which finitely many hills share the same height.

Consider hill H_1. To start, suppose that there exists a $t = S_1$ when

$$x_i(S_1) \in [p_1, B] \qquad \text{for all } i \in J_1^\infty. \tag{30}$$

Then

$$x_i(t) \in [p_1, B] \qquad \text{for all } i = 1, 2, \ldots, n \quad \text{and} \quad t \geqslant S_1. \tag{31}$$

This is true because, if some $x_i(t) = P_1$ at a time $t > S_1$, then $b(x_i(t)) = P_1 \geqslant b(x_j(t))$ for all $j \neq i$. Thus $M_i(t) = M(t) \geqslant 0$ and $\dot{x}_i(t) \geqslant 0$, which keeps $x_i(t) \in [p_1, B]$. Given that (31) holds, we now show that either jumps occur only among $i \in J_1^\infty$, while $x_i \in D_1$, or there exists a $t = T_1$ such that

$$b(x_i(t)) \leqslant P_2 \qquad \text{for all } i = 1, 2, \ldots \quad \text{and} \quad t \geqslant T_1. \tag{32}$$

This alternative is true because no jump can occur from an $i \in J_1^\infty$ to any hill H_j, $j \neq 1$, while $b(x_i) \geqslant P_2$. Since the $i \in J_1^\infty$ are *persistent* jump variables, either jumps continue to occur among the $i \in J_1^\infty$ on D_1, or eventually $y(t)$ enters the set $\bigcup_{j\geqslant 2} H_j$. In the former case, the heights $b(x_i)$ at successive jump times are monotone decreasing, since $b'(w) \leqslant 0$ for $w \in D_1$, and whenever $M_i(t) = M(t) \geqslant 0$, $\dot{x}_i(t) \geqslant 0$. Thus there exists a limiting height b_1^* to which the jump heights converge. In the latter case, there exists some time $t = T_1$ at which $y(T_1) \in \bigcup_{j\geqslant 2} H_j$. Consequently

$$b(x_i(T_1)) \leqslant P_2 \qquad \text{for all } i = 1, 2, \ldots, n. \tag{33}$$

It follows from (33) that (32) holds. To see this, suppose that time $t = U_1$ is the first time that some $b(x_i(t)) = P_2$. Then $M_i(U_1) = M(U_1) \geqslant 0$, so that $\dot{x}_i(U_1) \geqslant 0$. Moreover, by (30), either $x_i(U_1) = p_2$ or $x_i(U_1) \in D_1$, so that $b'(x_i(U_1)) \leqslant 0$. In both cases, $(d/dt)\, b(x_i(U_1)) = b'(x_i(U_1))\, \dot{x}_i(U_1) \leqslant 0$, which proves (33).

To summarize the above argument: If (30) holds at some $t = S_1$, then either b_1^* exists, or there exists a $t = T_1 \geqslant S_1$ such that (32) holds. It will be seen below how to complete the proof if b_1^* exists. Hence suppose that (31) and (32) hold.

Now consider H_2 at times $t \geqslant T_1$. Suppose that there exists a $t = S_2 \geqslant T_1$ such that

$$x_i(S_2) \in [p_2, B] \qquad \text{for all } i \in J_2^\infty. \tag{34}$$

Then

$$x_i(t) \in [p_2, B] \qquad \text{for all } i = 1, 2,..., n \quad \text{and} \quad t \geqslant S_2. \tag{35}$$

Property (35) follows from (34), since if some $x_i(t) = p_2$ at a time $t > S_2$, then by (32) $b(x_i(t)) = P_2 \geqslant b(x_j(t))$ for all $j \neq i$. Thus $M_i(t) = M(t) \geqslant 0$ and $\dot{x}_i(t) \geqslant 0$, which keeps $x_i(t) \in [p_2, B]$. From properties (31) and (35), we will conclude that either a limiting height b_2^* analogous to b_1^* exists, or there exists a time $t = T_2$ such that

$$b(x_i(t)) \leqslant P_3 \qquad \text{for all } i = 1, 2,..., n \quad \text{and} \quad t \geqslant T_2. \tag{36}$$

This alternative is true, because by (31) and (35), so long as jumps occur at heights that exceed P_3, they can only occur in $D_1 \cup D_2$. The heights at successive jump times are then monotone decreasing, whence b_2^* exists. If a jump occurs at a height $\leqslant P_3$ at some time $t = T_2$, then (36) holds. This is seen by considering the first time $t = U_2$ at which some $b(x_i(U_2)) = P_3$. Then $M_i(U_2) = M(U_2) \geqslant 0$, and by (32) and (35), either $x_i(U_2) = p_3$ or $x_i(U_2) \in D_1 \cup D_2$. In both cases, $(d/dt)\, b(x_i(U_2)) \leqslant 0$, thereby proving (36).

This argument can now be continued on hills of successively shorter heights to derive the following alternative after considering hill H_m: Either a limiting height b_m^* exists, or there exists a $t = T_m$ such that

$$b(x_i(t)) \leqslant P_{m+1} \qquad \text{for all } i = 1, 2,..., n \quad \text{and} \quad t \geqslant T_m. \tag{37}$$

The argument is continued until we reach the first hill H_r (possibly $r = 1$) on which there is no time at which all $x_i(t) \in [p_r, B]$ for all $i \in J_r^\infty$. In this case, we will conclude that there exists a time $t = S_r$ such that, for each $i = 1, 2,..., n$,

$$x_i(t) \in [0, p_r] \qquad \text{for } t \geqslant S_r \tag{38a}$$

or

$$x_i(t) \in [p_r, B] \qquad \text{for } t \geqslant S_r. \tag{38b}$$

This follows from the fact that if $x_i(S_r) \in [p_r, B]$, then $x_i(t) \in [p_r, B]$ for $t \geqslant S_r$. This conclusion is due to (37), since if $x_i(S_r) = p_r$ with $S_r \geqslant T_{r-1}$, then $M_i(S_r) = M(S_r) \geqslant 0$ and thus $\dot{x}_i(S_r) \geqslant 0$.

Given that (38) is true, it follows that either a limiting height b_r^* exists, or there exists a $t = T_r$ such that

$$b(x_i(t)) \leqslant P_{r+1} \qquad \text{for } i = 1, 2,..., n \quad \text{and} \quad t \geqslant T_r. \tag{39}$$

This can be shown as follows. Consider any jump sequence that starts with $y(t) \in A_r$ and $b(y(t)) > P_{r+1}$ at some time $t \geqslant S_r$. By (37) and (38), if such a jump sequence does not terminate, then $y(t) \in A_r$ at all future times. This is because, by (38), the jump variables on A_r cannot cross to D_r, and the only jump variables with $b(x_i(t)) > P_{r+1}$ that are not on A_r are on $\bigcup_{j=1}^r D_j$, and no jump can go from an ascending slope to a descending slope, by (24). If the jump sequence does not terminate, then a limiting height exists since the successive jump heights are monotone increasing on A_r. If the jump sequence does terminate, then there is no persistent jump variable $x_i(t) \in A_r$ such that $b(x_i)) > P_{r+1}$ after some time elapses. In this latter case, continue the argument on the set $\bigcup_{j=1}^r D_j$ to show that either a limiting height exists, or (39) holds.

Now consider hill H_{r+1}. It is clear that the above argument can be repeated to conclude that either a limiting height exists, or there exists a $t = T_{r+1}$ such that

$$b(x_i(t)) \leqslant P_{r+2} \qquad \text{for all } i = 1, 2,..., n \quad \text{and} \quad t \geqslant T_{r+1}. \tag{40}$$

If there are only finitely many hills on the interval $[0, B]$, then the above argument can be applied to each successively shorter hill until all hills are exhausted. Thus, after a finite amount of time goes by, either a limiting height exists, or no jump variable can cross a peak. In the latter case, each variable eventually gets trapped in a "bowl" between a contiguous descending slope and ascending slope.

Consider the case in which all variables eventually get trapped in their bowls. After this happens, what kinds of jumps can occur? Jumps can occur among descending slopes. The successive jumps then occur at successively lower heights. If this goes on indefinitely, then a limiting height exists. Jumps could instead occur among ascending slopes. The jumps then occur at successively higher heights. If this goes on indefinitely, then a limiting height again exists. Finally jumps can occur first among descending slopes until a jump occurs to some ascending slope. Thereafter only jumps among ascending slopes can occur, because no jump variable can continuously cross over a peak from an ascending slope to a descending slope, and no jump from an ascending slope can ever occur to a descending slope. In all cases, after a finite amount of time goes by, the successive jump heights converge monotonically to a limiting height. Indeed $b(y(t))$ is monotonic at all large times.

III. To complete the proof, given that only finite many hills exist, we must consider the case in which a limiting height is approached either on ascending slopes only, or on descending slopes only. Suppose for definiteness that there is a jump series among ascending slopes as $t \to \infty$; then $b(y(t))$ is monotone nondecreasing, and the successive heights at which jumps occur monotonically increase to b^*. We will now use this fact to prove that $\lim_{t\to\infty} c(x(t))$ exists and equals b^*. Then the proof can be completed by using Lemma 1 below.

The argument proceeds by contradiction. Suppose that $c(x(t))$ does not converge to b^*. Since $M(t) \geqslant 0$, it follows that

$$b^* \geqslant b(y(t)) \geqslant c(x(t)). \tag{41}$$

Thus there exists an $\epsilon > 0$ and a sequence of increasing times $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ such that

$$b^* - 3\epsilon \geq c(x(t_k)), \qquad k = 1, 2, \ldots \tag{42}$$

Since $c(x(t))$ is uniformly continuous for $t \geq 0$, there exists a $\beta > 0$ and a sequence of nonoverlapping intervals $[Y_k, Z_k]$ such that $Z_k - Y_k \geq \beta$ for $k = 1, 2, \ldots$ and

$$b^* - 2\epsilon \geq c(x(t)), \qquad t \in \bigcup_{k=1}^{\infty} [Y_k, Z_k]. \tag{43}$$

Given any k, choose a j such that $y(Y_k) = x_j(Y_k)$. Denote this j by $j = J(k)$; that is, $y(Y_k) = x_{J(k)}(Y_k)$. For k sufficiently large and $j = J(k)$,

$$b^* - b(x_j(Y_k)) \leq \epsilon/2. \tag{44}$$

Each function $b(x_j(t))$ is uniformly continuous in $t \geq 0$ for the same reason that $c(x(t))$ is. Consequently there exists a γ, $0 < \gamma \leq \beta$, such that for *all* large k and $j = J(k)$,

$$0 \leq b^* - b(x_j(t)) \leq \epsilon \quad \text{if} \quad t \in [Y_k, Y_k + \gamma]. \tag{45}$$

Thus by (43) and (45),

$$b(x_j(t)) - c(x(t)) \geq \epsilon, \qquad t \in [Y_k, Y_k + \gamma]. \tag{46}$$

Since also each $x_j(t)$ is bounded away from zero for $t \in [Y_k, Y_k + \gamma]$, (15) implies that there exists a $\delta > 0$ such that

$$a_j(x(t)) \geq \delta, \qquad t \in [Y_k, Y_k + \gamma]. \tag{47}$$

Putting (46) and (47) together shows, by (1), that

$$\dot{x}_j(t) \geq \delta\epsilon, \qquad t \in [Y_k, Y_k + \gamma], \tag{48}$$

or that

$$x_j(t) \geq x_j(Y_k) + \delta\epsilon(t - Y_k), \qquad t \in [Y_k, Y_k + \gamma]. \tag{49}$$

Thus $x_j(t)$ increases at an at least linear rate, that is independent of k, across an interval of length at least $\delta\epsilon\gamma$, when $t \in [Y_k, Y_k + \gamma]$. Denote this interval by Q_k.

Now choose an x_j such that $j = J(k)$ at infinitely many values of k. This x_j traverses infinitely many of the intervals Q_k. However, x_j is bounded for all $t \geq 0$. Thus the sets Q_k such that $j = J(k)$ must overlap as $k \to \infty$. More precisely, there exists an infinite subsequence $k_1 < k_2 < k_3 < \cdots$ of k's such that $j = J(k)$ and the closed interval $R_j = \bigcap_{m=1}^{\infty} Q_{k_m}$ has positive length. Since $\lim_{t\to\infty} b(y(t)) = b^*$, (45) holds for *any* $\epsilon > 0$ if k is chosen sufficiently large Consequently given any $\epsilon > 0$,

$$0 \leq b^* - b(w) \leq \epsilon \quad \text{if} \quad w \in R_j. \tag{50}$$

That is, $b(w) = b^*$ if $w \in R_j$. Thus at any sufficiently large time Y_k such that $j = J(k)$, it follows that

$$b(y(Y_k)) = b(x_j(Y_k)) = b^*. \tag{51}$$

In other words, the maximal variable attains the maximal height b^* at a *finite time* Y_k. Thereafter, no further jumps can occur, since $b(y(t))$ is monotone increasing. Thus all jumps cease after a finite time, and the proof can be completed as in Case 1, which shows that, indeed, $\lim_{j\to\infty} c(x(t)) = b^*$. This completes the proof when all $b_i \equiv b$, except for the application of Lemma 1.

Given arbitrary b_i, the same proof goes through because the original proof is "local" and "exhaustive". By this we mean the following. The proof when all $b_i \equiv b$ orders the hills by height and considers each hill in its turn. The position in $[0, B]$ of any given hill relative to other hills in the graph is immaterial. One simply has to define sets of persistent jump variables *on the hill*. In this sense, the proof is "local", because one can worry about one hill at a time. The proof is "exhaustive", because by considering all the hills, one can prove either that a limiting height exists, or that the variables eventually get trapped in their bowls. The latter case then also implies that a limiting height exists. When the behavior near the limiting height is considered, all that matters is that each limiting variable is on an ascending slope, or that each limiting variable is on a descending slope. *Where* these slopes are to be found is irrelevant—again, a "local" condition is studied.

To prove the theorem given arbitrary b_i, it suffices to consider all the hills on the graphs of all the b_i. Order these hills by height, and then proceed exactly as in the $b_i \equiv b$ case. After one exhausts all these hills, one automatically exhausts all the hills in the graph of each b_i, so the proof can be completed in the same fashion.

Now we indicate how the above arguments can be adapted to cases in which there are infinitely many hills on $[0, B]$. Order the hills so that $P_1 \geq P_2 \geq P_3 \cdots$. By hypothesis $b(w)$ is either continuous on $[0, B]$, hence uniformly continuous; or $b(0) = \infty$ and $b(w)$ is uniformly continuous on $[\delta, B]$ no matter how small $\delta > 0$ is chosen. Thus given any interval $[\delta, B]$ and any $\epsilon > 0$, there exists in $[\delta, B]$ only finitely many hills H_i such that $P_i - V_i \geq \epsilon$.

First consider the case in which no height P_k or depth V_k is a limit point of other heights or depths (including the case where $P_1 = +\infty$). Then $\lim_{k\to\infty}(P_k - V_k) = 0$. Since also the sequence $\{P_k\}$ is monotone decreasing and bounded below, $\lim_{k\to\infty} P_k = \lim_{k\to\infty} V_k$. Consequently, as $t \to \infty$ either the jumps are restricted to finitely many hills, which can be treated as above, or (40) holds for all $r \geq 0$, and

$$\lim_{t\to\infty} b(y(t)) = \lim_{k\to\infty} P_k. \tag{52}$$

By(52), a limiting height exists, and certain x_i must keep moving onto shorter hills as $t \to \infty$. Since these hills approach a limiting width of zero, and since the x_i are continuous, certain limits $x_i(\infty)$ exist. The proof can now be continued by adapting the argument in (25)–(29). The only change is that (27) holds because $x_i(t)$ is forced away from zero by being driven onto ever shorter hills. Given the existence of $c(x(\infty))$, it is proved that the limits of the remaining x_i exist by using the fact that no P_k or V_k is a limit point to drive all these x_i onto a definite ascending or descending slope, but not a peak or valley, as $t \to \infty$.

Now consider a case in which the maximal height P_1 is a limit point of other heights. Given an infinite sequence $\{w_k\}$ such that P_1 is a limit point of $\{b(w_k)\}$, then a cluster point $\bar{w}$ of $\{w_k\}$ exists and satisfies $b(\bar{w}) = P_1$ due to the continuity of $b(w)$. Consider the case in which finitely many such cluster points $\bar{w}_{11}, \bar{w}_{12}, \ldots, \bar{w}_{1r_1}$ exist. By the uniform continuity of $b(w)$, each $\bar{w}_{1k}$ is a limit of hills that become arbitrarily small as $w \to \bar{w}_{1k}$. Intuitively speaking, $\bar{w}_{1k}$ is the peak of "a hill with infinitely small wiggles" near $\bar{w}_{1k}$. In particular, the depths of the hills close to any $\bar{w}_{1k}$ approach P_1 at a uniform rate as $w \to \bar{w}_{1k}$. Thus if $b(y(t))$ gets close to P_1 in value, it remains close to P_1 in value unless $y(t)$ eventually crosses some $\bar{w}_{1k}$ value and descends a sequence of hills. More precisely, either $\lim_{t\to\infty} b(y(t)) = P_1$ or there exists an $\epsilon > 0$ and a T_ϵ such that $b(x_i(t)) \leqslant P_1 - \epsilon$ for $i = 1, 2, \ldots, n$ and all $t \geqslant T_\epsilon$. In the latter case, let P_2 be the maximal hill height $\leqslant P_1 - \epsilon$ among all the hills on which $y(t)$ sits at some $t \geqslant T_\epsilon$. Repeat the above argument, assuming that there are finitely many cluster points $\bar{w}_{21}, \bar{w}_{22}, \ldots, \bar{w}_{2r_2}$ such that $b(\bar{w}_{2k}) = P_2$. Again either $\lim_{t\to\infty} b(y(t)) = P_2$ or there exists an $\epsilon > 0$ and a T_ϵ such that $b(x_i(t)) \leqslant P_2 - \epsilon$ for $i = 1, 2, \ldots, n$ and $t \geqslant T_\epsilon$. This argument is now repeated iteratively. The main point is that finitely many cluster point peaks can be treated like finitely many peaks without cluster points. In the case where infinitely many cluster points to a given height exist, one notes that only in regions where these cluster points are isolated can a jump variable possibly escape to a distinct height; in effect, since only finitely many $P_k - V_k$ values are not smaller than any prescribed $\epsilon > 0$, the argument can again be reduced essentially to the finite case. More precisely, given any limiting height b^*, use the boundedness and uniform continuity of b_j to cover all cluster points of b_j with finitely many intervals in which $b^* \geqslant b_j \geqslant b^* - \epsilon$. These intervals surround the isolated cluster points of b_j as well as finitely many cluster points around which other cluster points of b_j cluster. Using such a finite covering at every stage of the argument, argue as in the case of finitely many cluster points to conclude that the limit $b^* = \lim_{t\to\infty} b_{I(t)}(y(t))$ exists in the general case.

It remains only to prove that $\lim_{t\to\infty} c(x(t))$ exists and equals b^*, since then the proof can be completed using Lemma 1. For convenience, the notation $B(t) = b_{I(t)}(y(t))$ will be used to denote the maximum height at any time. The proof proceeds by supposing that $c(x(t))$ does not approach b^*. Then (42)–(51) follow as before. In particular, there exists a j and a sequence of times $t_m = Y_{k_m}$ with $\lim_{m\to\infty} t_m = \infty$ such that

$$b_j(x_j(t_m)) = B(t_m) = b^*, \qquad m = 1, 2, \ldots \tag{53}$$

The main new difficulty is that the x_i's need not be trapped on only descending slopes or only ascending slopes, so $B(t)$ is not necessarily monotonic at all large times; that is, $b^* - B(t)$ can change sign at arbitrarily large times. Suppose that this happens. Then there exists a sequence $\{s_m\}$ with $t_m < s_m < t_{m+1}$ such that

$$B(s_m) > b^*, \qquad m = 1, 2, \ldots \tag{54}$$

despite the fact that (53) holds. How can this happen? By (53) and (54), $B(t_m) < B(s_m)$ and $B(t_{m+1}) < B(s_m)$ despite the fact that $t_m < s_m < t_{m+1}$. In order for the maximal height to increase and then decrease, *some* variable x_i must go over a hill: The increase requires the maximal variable to be on an ascending slope, and the decrease requires the maximal variable to be on a descending slope; since no jump can occur from an ascending slope to a descending slope, the maximal variable must go over a hill. Moreover, since $\lim_{t\to\infty} B(t) = b^*$, given any $\epsilon > 0$ there exists a T_ϵ such that

$$b^* + \epsilon \geqslant B(t) \qquad \text{for all } t \geqslant T_\epsilon. \tag{55}$$

Consequently, the maximal variable cannot go over a hill whose height exceeds $b^* + \epsilon$ after time $t = T_\epsilon$. This forces the maximal variable to go over infinitely many hills as $t \to \infty$, since no matter how much higher the hill is than b^*, after some finite time goes by, ϵ can be chosen so small that the hill is too high to be the one that the maximal variable goes over after that time. Since there are only finitely many hills whose width and depth exceed any fixed size $\delta > 0$, by waiting sufficiently long, the maximal variable is driven to hills of arbitrarily small width and depth. Now choose an i such that $b_i(x_i(s_m)) = B(s_m)$ at infinitely many values of m. The above argument shows that $x_i(t)$ eventually gets driven onto, and trapped within, arbitrarily small hills as $t \to \infty$. Consequently, the limit $x_i(\infty)$ exists and $b_i(x_i(\infty)) = b^*$. The argument in Case 1 can now easily be adapted to prove that $\lim_{t\to\infty} c(x(t)) = b^*$.

In all cases, it has now been proved that

$$\lim_{t\to\infty} c(x(t)) = b^*. \tag{29}$$

This fact is now used to complete the proof using the following Lemma.

LEMMA 1. *If system* (1) *obeys properties* (I)–(IV), *and limit* (29) *exists, then the limits* $b_i(x_i(\infty))$ *exist and equal* $c(x(\infty))$, $i = 1, 2, \ldots, n$. *If, moreover, the oscillation condition does not hold, then the limits* $x_i(\infty)$ *exist*, $i = 1, 2, \ldots, n$.

Proof. By (29), given any $\epsilon > 0$, there exists a T_ϵ such that

$$|c(x(\infty)) - c(x(t))| < \epsilon \qquad \text{if} \qquad t \geqslant T_\epsilon. \tag{56}$$

Suppose that

$$b_i(x_i(t)) > c(x(\infty)) + \epsilon \tag{57}$$

at some time $t \geqslant T_\epsilon$. Then, by (56), $\dot{x}_i(t) > 0$. Consequently, $x_i(t)$ monotonically increases towards the limit $x_i(\infty)$ unless there is a time $t \geqslant T_\epsilon$ at which

$$b_i(x_i(t)) = c(x(\infty)) + \epsilon. \tag{58}$$

There must be such a time, since otherwise $b_i(x_i(\infty)) - c(x(\infty)) \geqslant \epsilon$, and thus $x_i(\infty) = \infty$, which is impossible. Consider the first time $t = T \geqslant T_\epsilon$ at which (58) holds after an interval of times during which (57) holds. At this time,

$$0 \geqslant \frac{d}{dt} b_i(x_i(T)) = b_i'(x_i(T))\, \dot{x}_i(T). \tag{59}$$

By (56), $\dot{x}_i(T) \geqslant 0$, and thus $b'(x_i(T)) \leqslant 0$, so that $x_i(T)$ is on a descending slope, or plateau, of b_i. Because (56) holds for all $t \geqslant T$, it follows that $x_i(t) \geqslant x_i(T)$ for all $t \geqslant T$, since whenever $x_i(t) = x_i(T)$, (56) and (58) imply that $\dot{x}_i(t) \geqslant 0$. The above argument can now be iterated. After (57) holds during some time interval, there must be a time when (58) holds while x_i is on a descending slope or plateau. Letting T_{ij} be the jth time at which (58) holds after (57) holds, it follows that $x_i(t) \geqslant x_i(T_{ij})$ for $t \geqslant T_{ij}$, and that $x_i(T_{i1}) < x_i(T_{i2}) < \cdots$. If there are only finitely many T_{ij}'s, then there exists a time U_ϵ such that

$$b_i(x_i(t)) \leqslant c(x(\infty)) + \epsilon \qquad \text{for all } t \geqslant U_\epsilon. \tag{60}$$

Otherwise, there must exist infinitely many hills in the graph of b_i. Since x_i is bounded, and bounded away from 0, and b_i is continuous on the compact set within which x_i fluctuates, b_i is also uniformly continuous. Consequently, given any δ, there exist only finitely many hills in the graph of b_i whose width or depth is greater than δ. If x_i traverses infinitely many hills on which (58) holds after (57) holds, it is eventually forced onto arbitrarily small hills whose heights P_k and depths V_k both approach $c(x(\infty)) + \epsilon$, by (58). Thus at *all* large times, $b_i(x_i(t)) - c(x(t)) \geqslant \epsilon/2$, which again implies the impossible conclusion that $x_i(\infty) = \infty$. Consequently, given any $\epsilon > 0$, there exists a time U_ϵ such that (60) holds. A similar argument with reversed inequalities allows us to conclude that there exists a time V_ϵ such that

$$b_i(x_i(t)) \geqslant c(x(\infty)) - \epsilon \qquad \text{for all } t \geqslant V_\epsilon. \tag{61}$$

Since both (60) and (61) hold for any $\epsilon > 0$,

$$\lim_{t\to\infty} b_i(x_i(t)) = c(x(\infty)). \tag{62}$$

The same argument holds for all $i = 1, 2,..., n$. Consequently, system (1) achieves weak global consensus. Moreover, since all the limits $b_i(x_i(\infty))$ exist and equal $c(x(\infty))$, it follows that all the limits $\dot{x}_i(\infty)$ exist and equal 0. Thus whatever oscillations occur in the $x_i(t)$ become arbitrarily slow as $t \to \infty$.

From (62), it follows that the limit $x_i(\infty)$ exists unless there exists a nontrivial interval W_i of values throughout which $b_i(w) = b^*$. If $x_i(t) \in W_i$ at all large times, then $x_i(t)$ might oscillate back and forth across W_i as $t \to \infty$ without contradicting the fact that $b_i(x_i(\infty)) = b^* = c(x(\infty))$. By (1), $x_i(t)$ can oscillate back and forth across W_i at arbitrarily large times only if $c(x(t))$ oscillates above and below b^* at arbitrarily large times. Since $M(t) = B(t) - c(x(t)) \geqslant 0$, $c(x(t))$ can only oscillate above b^* at arbitrarily large times if $B(t)$ also exceeds b^* at arbitrarily large times. In particular, if $B(t)$ gets trapped on ascending slopes at all large times, then this cannot happen, since then $B(t)$ monotonically increases to b^* while $B(t) \geqslant c(x(t))$. Consequently, all limits $x_i(\infty)$ exist in this case.

Moreover, if every b_i has only finitely many hills above the height b^*, then again all limits $x_i(\infty)$ exist. To see this, wait until all the x_i have crossed over all the peaks of those hills that they shall ever cross. Suppose that this occurs before time $t = T$. Because $b_i(x_i(\infty)) = b^*$, the following alternative holds: Either $b_i(x_i(t)) \geqslant b^*$ for all $t \geqslant T$, or $b_i(x_i(t)) < b^*$ for all $t \geqslant T$. Only those $x_i(t)$ for which $b_i(x_i(t)) \geqslant b^*$ can ever equal $y(t)$ at arbitrarily large times. Henceforth we restrict attention to these persistent jump variables. Since all persistent $x_i(t)$ have crossed their last hill before time $t = T$, and $b_i(x_i(t)) \geqslant b^* = b_i(x_i(\infty))$, it follows that all persistent $x_i(t)$ are on descending slopes for $t \geqslant T$; that is, $b_i'(x_i(t)) \leqslant 0$ for $t \geqslant T$. Using this fact, we will prove that $\int_0^\infty M(t)\, dt < \infty$. This latter inequality implies that all limits $x_i(\infty)$ exist; see [11, Theorem 1].

Since $y(t)$ is restricted to descending slopes for all $t \geqslant T$, $B(t)$ is monotone decreasing for $t \geqslant T$. Consider the trajectory of a given $x_i(t)$ at all times when $y(t) = x_i(t)$, $t \geqslant T$. Suppose for definiteness that there is a sequence U_{i1}, U_{i2},... of nonoverlapping intervals of time, whose union is U_i, such that $y(t) = x_i(t)$ only if $t \in U_i$. Suppose moreover that $U_{ik} = [S_{ik}, T_{ik})$. Whenever $y(t) = x_i(t)$, it follows that $\dot{x}_i(t) \geqslant 0$. Consequently $x_i(T_{ik}) \geqslant x_i(S_{ik})$. It is also true that $x_i(S_{i,k+1}) \geqslant x_i(T_{ik})$. This follows from the fact that x_i is trapped on a descending slope, and that $\dot{B}(t) \leqslant 0$ for all $t \geqslant T$. Thus the nonoverlapping intervals of time U_{ik} generate nonoverlapping intervals $[x_i(S_{ik}), x_i(T_{ik}))$ in the range of x_i. Since x_i is bounded, the total length of these intervals, namely $\Sigma_k[x_i(T_{ik}) - x_i(S_{ik})]$, is finite. This total length can also be written as $\int_{U_i} \dot{x}_i\, dt$, which can be written as $\int_{U_i} a_i M\, dt$. Since each x_i is bounded away from zero, it follows that $\int_{U_i} M\, dt$ is finite for every persistent x_i. However, $\int_T^\infty M\, dt$ is the sum over a finite number of these integrals, and thus $\int_0^\infty M\, dt < \infty$.

Each limit $x_j(\infty)$ therefore exists unless b_j has infinitely many hills H_1, H_2,... whose peak heights P_1, P_2,... exceed b^* and $\lim_{m\to\infty} P_m = b^*$. Moreover, $x_j(t)$ must reach each of these hills as $t \to \infty$, and $x_j(t) = y(t)$ for some time at which $x_j(t)$ is on each hill. Otherwise $y(t)$ would get trapped on descending slopes at all large times. Also, by the uniform continuity of b_j, the depths V_1, V_2,... of these hills also satisfy $\lim_{m\to\infty} V_m = b^*$, and there exists a w^* such that the peaks and valleys of the hills converge to w^* as $m \to \infty$.

First consider the case in which $b_j(x_j(t)) \geqslant b^*$ at all large times, despite the fact that b_j has infinitely many hills. Consider times $t = T$ at which $c(x(T)) = b^*$. There must exist infinitely many such times, approaching infinity, at which $\dot{c}(x(T)) < 0$, so that $c(x(t))$ can oscillate around b^* infinitely often. At every such $t = T$, some variable, say x_j, satisfies $y(T) = x_j(T)$. Thus $\dot{x}_j(T) \geqslant 0$. Moreover the x_i for which $b_i(x_i(T)) = b^*$ satisfies $\dot{x}_i(T) = 0$. Since

$$\dot{c}(x(T)) = \sum_{m=1}^{n} \frac{\partial c}{\partial x_m}(x(T))\, \dot{x}_m(T)$$

where all

$$\frac{\partial c}{\partial x_m}(x(T)) \geqslant 0,$$

there must exist an x_k, with $k \neq i, j$, such that $\dot{x}_k(T) < 0$ at infinitely many of the times T. This justifies the oscillation condition.

In the remaining case, there can exist b_j which oscillate above and below b^* on infinitely many hills. Then a similar argument holds: In order for x_j to get across infinitely many hills, there must exist infinitely many $t = T$, approaching infinity, at which $b_j(x_i(T)) > b^*$, $c(x(T)) = b^*$, and $\dot{c}(x(T)) < 0$. Since $\dot{x}_j(T) \geqslant 0$, there must exist an x_k such that $\dot{x}_k(T) < 0$ at infinitely many values of T. Thus all $x_i(\infty)$ exist, except possibly in those cases wherein the oscillation condition holds.

4. Finite Jump Condition

The proof of Theorem 1 does not rule out the possibility that infinitely many jumps occur, say if a limiting height exists. Theorem 1 of [12] describes systems of the form

$$\dot{x}_i = a(x)\, g(x_i)\, [b(x_i) - c(x)], \tag{63}$$

in which only finitely many jumps occur, and in which the jump trends through time can be analyzed. This theorem depends on two properties that do not generally hold in (1): First, because of the form of equation (63), the variables are *ordered* in time; that is, they can be labeled so that $x_1(t) \leqslant x_2(t) \leqslant \cdots \leqslant x_n(t)$ for $t \geqslant 0$. Second, a *self-similarity* condition is assumed to hold between the hills of the graph of $b(w)$. This condition requires that the highest hills of the graph are also the speepest hills. Self-similarity explicates the intuitive idea that each hill is due to averaging over some random factor that is distributed across a subpopulation of each population, and that the averaging process will automatically produce a correlation between the steepness and height of the hills in many cases. Theorem 1 above indicates that neither the ordering nor the self-similarity is necessary to produce global limits.

The question of when a system has only finitely many jumps is of considerable physical interest, since after all jumps cease the system has "decided" on its asymptotic pattern. There exist systems more general than (63) in which only finitely many jumps can occur even if self-similarity does not hold. In these systems, the infinite sequences of jumps towards a limiting height are ruled out by imposing a dominance condition on the possible jumps between slopes. This dominance condition is weaker than self-similarity because there need not be any relationship between the relative height and steepness of a hill.

Theorem 2 (Finite Jump Sequence). *Given any $n \geqslant 2$, consider the systems*

$$\dot{x}_i = a(x)\, g_i(x_i)\, [b_i(x_i) - c(x)], \tag{64}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $i = 1, 2, \ldots, n$. Let the following hypotheses hold:

1. *Smoothness:*

(a) *$a(x)$ is continuous for $x \geqslant 0$;*

(b) *$g_i(x_i)$ is continuous for $x_i \geqslant 0$;*

(c) *$b_i(x_i)$ is either continuous with piecewise derivatives for $x_i \geqslant 0$, or is continuous with piecewise derivatives for $x_i > 0$ and $b_i(0) = \infty$;*

(d) *$c(x)$ is continuous with piecewise derivatives for $x \geqslant 0$.*

2. *Nonnegativity:*

$$a(x) > 0 \quad \text{if} \quad x \geqslant 0, \tag{65}$$

$$g_i(x_i) > g_i(0) = 0, \quad x_i > 0, \tag{66}$$

and

$$\int_0^{\lambda} \frac{dw}{g_i(w)} = \infty. \tag{67}$$

3. *Boundedness:*

$$\limsup_{w\to\infty} b_i(w) < c(0, 0, \ldots, \infty, \ldots, 0, 0), \tag{68}$$

where "∞" is in the ith entry, $i = 1, 2, \ldots, n$.

4. *Competition:*

$$\frac{\partial c}{\partial x_k} \geqslant 0, \quad k = 1, 2, \ldots, n; \tag{69}$$

5. *Slope Dominance:*

Let there exist finitely many ascending slopes A_{ik} and descending slopes D_{ik} on the graph of each function $b_i(w)$, $w \in [0, B]$. Given any pair A_{jk} and A_{lm} of ascending slopes, let the slope functions $s_i(w) = g_i(w)\, b_i'(w)$ satisfy either

$$s_j(w_j) \geqslant s_l(w_l) \quad \text{if} \quad b_j(w_j) = b_l(w_l) \quad \text{and} \quad w_j \in A_{jk}, \quad w_l \in A_{lm} \tag{70a}$$

or

$$s_j(w_j) \leqslant s_l(w_l) \quad \text{if} \quad b_j(w_j) = b_l(w_l) \quad \text{and} \quad w_j \in A_{jk}, \quad w_l \in A_{lm}. \tag{70b}$$

Given any pair D_{jk} and D_{lm} of descending slopes, let the slope functions satisfy either

$$s_j(w_j) \geqslant s_l(w_l) \quad \text{if} \quad b_j(w_j) = b_l(w_l) \quad \text{and} \quad w_j \in D_{jk}, \quad w_l \in D_{lm} \tag{71a}$$

or

$$s_j(w_j) \leqslant s_l(w_l) \quad \text{if} \quad b_j(w_j) = b_l(w_l) \quad \text{and} \quad w_j \in D_{jk}, \quad w_l \in D_{lm}. \tag{71b}$$

Then given any nonnegative initial data $x(0)$, finite nonnegative limits $x(\infty)$ are approached after finitely many jumps occur.

Proof. The proof proceeds as in Theorem 1 until the case of a limiting height is considered. Jumps then occur between finitely many variables on (say) ascending slopes. By (24) and (64), a jump can occur from i to j at time T only if

$$s_j(x_j(T)) > s_i(x_i(T)) \quad \text{and} \quad b_i(x_i(T)) = b_j(x_j(T)). \tag{72}$$

Thus by (70), once a jump occurs from a variable on a given ascending slope to a different ascending slope, a jump can never return to the original variable. Since these are only finitely many variables, only finitely many jumps are possible, and the proof can be completed as in Case 1 of Theorem 1.

When Theorem 2 is applied to a generalized Volterra–Lotka system of the form

$$\dot{x}_i = D_i(x_i)\left[1 - \sum_{k=1}^{n} f_k(x_k)\, E_{ki}(x)\right], \tag{73}$$

with $E_{ki}(x) = F_k(x_k)\, G_i(x_i)$, the slope function

$$s_i(w) = \frac{-D_i(w)\, G_i'(w)}{G_i(w)}. \tag{74}$$

By (70) and (71), $s_j(w_j)$ and $s_l(w_l)$ are compared at values w_j and w_l such that $b_j(w_j) = b_l(w_l)$. Since $b_i(w) = G_i^{-1}(w)$ in this case, the relative sizes of the functions $S_j(w_j) = D_j(w_j)\, G_j'(w_j)$ and $S_l(w_l) = D_l(w_l)\, G'(w_l)$ must be compared at values of w_j and w_l such that $G_j(w_j) = G_l(w_i)$. This observation leads to the following corollary.

COROLLARY 3. *Let system (73) with $E_{ki}(x) = F_k(x)\, G_i(x_i)$ satisfy the conditions of Corollary 2. In addition, suppose that there exist finitely many ascending slopes A_{ik} and descending slopes D_{ik} of the functions $G_i(w)$, $w \in [0, B]$. Given any pair A_{jk} and A_{lm} of ascending slopes, let the slope functions $S_i(w) = D_i(w)\, G_i'(w)$ satisfy either*

$$S_j(w_j) \geqslant S_l(w_l) \quad \text{if} \quad G_j(w_j) = G_l(w_l) \quad \text{and} \quad w_j \in A_{jk}, \quad w_l \in A_{lm} \tag{75a}$$

or

$$S_j(w_j) \leqslant S_l(w_l) \quad \text{if} \quad G_j(w_j) = G_l(w_l) \quad \text{and} \quad w_j \in A_{jk}, \quad w_l \in A_{lm}. \tag{75b}$$

Given any pair D_{jk} and D_{lm} of descending slopes, let the slope functions satisfy either

$$S_j(w_j) \geqslant S_l(w_l) \quad \text{if} \quad G_j(w_j) = G_l(w_l) \quad \text{and} \quad w_j \in D_{jk}, \quad w_l \in D_{lm} \tag{76a}$$

or

$$S_j(w_j) \leqslant S_l(w_l) \quad \text{if} \quad G_j(w_j) = G_l(w_l) \quad \text{and} \quad w_j \in D_{jk}, \quad w_l \in D_{lm}. \tag{76b}$$

Then given any nonnegative initial data $x(0)$, finite nonnegative limits $x(\infty)$ are approached after finitely many jumps occur.

5. MAXIMIZING PREFERENCE AND CONTRAST

Since $b_i(x_i) = G_i^{-1}(x_i)$ in the Volterra–Lotka systems (73), local minima of G_i are local maxima of b_i. Thus the fact that dynamical boundaries are switched in earliest at the abscissas of the highest peaks of b_i translates into the fact that dynamical boundaries are switched in earliest at the lowest valleys of G_i. Each G_i can be interpreted as a preference function, since the vector function $G(x) = (G_1(x_1), G_2(x_2),..., G_n(x_n))$ rank-orders the strength of signals from any population v_k to all the populations $v_1, v_2,..., v_n$ when the system is in state x. Thus the above results proves that the dynamical boundaries are switched in at successively highly values of preference as $t \to \infty$. Once x_i crosses the lowest valleys of the preference function G_i, it can never cross them again. This defines a statistical tendency for the system to try to achieve the largest preference values that are compatible with its initial data $x(0)$ and the structure of the state-dependent preference order $G(x)$. Thus these Volterra–Lotka systems tend to maximize preference, just as the analogous neural networks (4) tend to maximize contrast, other things equal. It would appear to be wrong, however, to assume that a maximization principle could be used to express this trend in these nonstationary systems, although the search for such a principle is always a tempting adventure. Such a principle is often associated with a Liapunov function in classical examples. In the present examples, the maximum function $b(y(t))$ is not a Liapunov function at all values of $t \geqslant 0$. However, where only finitely many hills exist, $b(y(t))$ becomes a Liapunov function *after* all the dynamical boundaries have been laid down; that is, after all the decisions have already been made. This is true because $b(y(t))$ is then either restricted to descending slopes at all large times, or after one jump to an ascending slope, is restricted thereafter to ascending slopes. In the former case, $b(y(t))$ is a Liapunov function at large times; in the latter case, $-b(y(t))$ is a Liapunov function at large times. Thus, after the initially nonstationary dynamics of decision-making is over, the system then settles down towards a "classical limit". A similar trend occurs in learning networks; *after* the nonstationary phase of learning is over, the network settles down to a stationary memory phase, which is described by a stationary Markov chain [13]. Such examples suggest that global insights into the nonstationary processes suggested by biology require concepts and methods that genuinely transcend those that have proved so useful toward understanding essentially stationary phenomena.

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.