Reprint Series
6 March 1987, Volume 235, pp. 1226–1227

SCIENCE

Computing with Neural Networks

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Hopfield and Tank (*1*) refer to "A new concept for understanding the dynamics of neural circuitry" using the equation (in a slightly different notation)

$$C_i\frac{du_i}{dt} = -\frac{1}{R_i}u_i + \sum_{j=1}^{n} T_{ij} f_j(u_j) + I_i \quad (i = 1,\ldots, n) \qquad (1)$$

for the neuron state variables u_i. The concept is that the variables $u_i(t)$ approach equilibrium as $t \to \infty$ if the connections T_{ij} are symmetric ($T_{ij} = T_{ji}$). Hopfield and Tank also state that "a nonsymmetric circuit . . . has trajectories corresponding to complicated oscillatory behaviors . . . but as yet we lack the mathematical tools to manipulate and understand them at a computational level" (*1*, p. 629), and that "the symmetry of the networks is natural because, in simple associations, if A is associated with B, B is symmetrically associated with A" (*1*, p. 629).

Associations are often asymmetric, as in the asymmetric error distributions arising during list learning (*2*). Neural network models (*3*) explain these distributions when one uses Eq. 1 supplemented by an associative learning equation for the connections T_{ij}

$$\frac{dT_{ij}}{dt} = -AT_{ij} + Bu_i f_j(u_j) \qquad (2)$$

Because of the nonlinear term $u_i f_j(u_j)$ in Eq. 2, $T_{ij} \neq T_{ji}$.

Stability theorems (*4*) have been proved about neural networks which include and generalize Eqs. 1 and 2. Thus symmetry is not necessary to prove associative learning and memory storage by neural networks. Nor is symmetry needed to design stable neural networks for adaptive pattern recognition (*5*). Methods have also been developed (*6*) for analyzing the oscillatory behavior of neural circuits. We believe that the relation between symmetry and stability in neural networks is much more subtle and better understood than Hopfield and Tank (*1*) suggest.

Nonetheless, symmetry does help to analyze the system represented by Eq. 1. In fact, we (M.A.C. and S.G.) (*7*) independently discovered an energy function for neural networks "designed to transform and store a large variety of patterns. Our analysis includes systems which possess infinitely many equilibrium points" (*7*, p. 818), examples of which have been constructed (*8*). These networks are

$$\frac{du_i}{dt} = a_i(u_i)\left[b_i(u_i) - \sum_{j=1}^{n} c_{ij} d_j(u_j)\right] \quad (i = 1, \ldots, n) \qquad (3)$$

Given symmetric connections ($c_{ij} = c_{ji}$), the energy function is

$$V = -\sum_{i=1}^{n} \int_0^{u_i} b_i(\xi_i) d_i'(\xi_i) d\xi_i + \frac{1}{2}\sum_{j,k=1}^{n} c_{jk} d_j(u_j) d_k(u_k) \qquad (4)$$

Along system trajectories

$$\frac{d}{dt}V = -\sum_{i=1}^{n} a_i(u_i) d_i'(u_i)\left[b_i(u_i) - \sum_{k=1}^{n} c_{ik} d_k(u_k)\right]^2 \qquad (5)$$

If $a_i(u_i) \geq 0$ and $d_i'(u_i) \geq 0$, then $\frac{d}{dt}V \leq 0$, which is the key property of an energy function. We (M.A.C. and S.G.) have noted that "the simpler additive neural networks . . . are also included in our analysis" (*7*, p. 819). The system represented by Eq. 3 reduces to the additive network (Eq. 1) when $a_i(u_i) = C_i^{-1}, b_i(u_i) = -1/R_i \; u_i + I_i$, $c_{ij} = -T_{ij}$ and $d_j(u_j) = f_j(u_j)$. Then

$$V = \sum_{i=1}^{n} \frac{1}{R_i}\int_0^{u_i} \xi_i f_i'(\xi_i) d\xi_i - \sum_{i=1}^{n} I_i f_i(u_i) - \frac{1}{2}\sum_{j,k=1}^{n} T_{jk} f_j(u_j) f_k(u_k) \qquad (6)$$

which includes the energy functions used in (*1*). We (M.A.C. and S.G.) (*7*) also analyzed the more difficult and physiologically important cases where the cells obey membrane, or shunting, equations and the signal functions $d_j(u_j)$ may have output thresholds.

Thus we consider the "new concept" in (*1*) to be a recent special case of an established neural network theory (*7, 8, 9*).

Hopfield and Tank also assert that "Unexpectedly, new computational properties resulted . . . from the use of nonlinear graded-response neurons instead of the two-state neurons of the earlier models" (*1*, p. 625). It has long been understood that two-state neuronal models differ computationally from graded-response models with sigmoid signal functions (*6, 8, 10*).

The application of neural network theory to technology would be expedited by further consideration of known results (*11*).

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12. Supported in part by the Air Force Office of Scientific Research (AFOSR 85-0149 and AFOSR F 49620-86-C-0037), the Army Research Office (ARO DAAG-29-85-K-0095), and the National Science Foundation (NSF DMS-84-13119 and NSF IST-84-17756). We thank C. Suchta and C. Yanakakis for their valuable assistance in the preparation of the manuscript.

14 August 1986; accepted 5 December 1986