Problem Set 5

(Lagrange Multiplier Methods)

Due: Wednesday, March 21, 2018

Problem 1

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{Q}$ is an $n \times n$ positive semidefinite matrix. We introduce the logarithmic barrier problem:

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} - \mu \sum_{j=1}^{n} \log x_j \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned}$$

(a) Show that the associated optimality conditions are:

$$\begin{aligned} \mathbf{A}\mathbf{x}(\mu) &= \mathbf{b}, \\ -\mathbf{Q}\mathbf{x}(\mu) + \mathbf{A}'\mathbf{p}(\mu) + \mathbf{s}(\mu) &= \mathbf{c}, \\ \mathbf{X}(\mu)\mathbf{S}(\mu)\mathbf{e} &= \mathbf{e}\mu, \end{aligned}$$

where $\mathbf{X}(\mu) = \text{diag}\big(x_1(\mu), \ldots, x_n(\mu)\big)$ and $\mathbf{S}(\mu) = \text{diag}\big(s_1(\mu), \ldots, s_n(\mu)\big)$.

(b) Show that a Newton direction can be found by solving the following system of equations.

$$\begin{aligned} \mathbf{A}\mathbf{d}_x^k &= \mathbf{0}, \\ -\mathbf{Q}\mathbf{d}_x^k + \mathbf{A}'\mathbf{d}_p^k + \mathbf{d}_s^k &= \mathbf{0}, \\ \mathbf{S}_k\mathbf{d}_x^k + \mathbf{X}_k\mathbf{d}_s^k &= \mu^k\mathbf{e} - \mathbf{X}_k\mathbf{S}_k\mathbf{e}. \end{aligned}$$

(c) Develop a primal-dual path following interior point algorithm. Implement the algorithm in Matlab and test it in the following example

$$Q = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}, \qquad c = (2, 1), \qquad A = [1 \;\; 1], \qquad b = 1.$$

Report the result and plot the evolution of the algorithm.

Remark 1 *You might want to consult the book by Bertsimas and Tsitsiklis, "Introduction to Linear Optimization", where a primal-dual algorithm is given for the linear programming case, and some implementation issues are discussed.*

Problem 2: Exercise 5.2.1 of [Ber]

You can skip part (c).