Problem 1

Consider the optimization problem

\[
\begin{align*}
\text{minimize} & \quad c'x + \frac{1}{2}x'Qx \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

where \( Q \) is an \( n \times n \) positive semidefinite matrix. We introduce the logarithmic barrier problem:

\[
\begin{align*}
\text{minimize} & \quad c'x + \frac{1}{2}x'Qx - \mu \sum_{j=1}^{n} \log x_j \\
\text{subject to} & \quad Ax = b.
\end{align*}
\]

(a) Show that the associated optimality conditions are:

\[
\begin{align*}
Ax(\mu) &= b, \\
-Qx(\mu) + A'p(\mu) + s(\mu) &= c, \\
X(\mu)S(\mu)e &= e\mu,
\end{align*}
\]

where \( X(\mu) = \text{diag}(x_1(\mu), \ldots, x_n(\mu)) \) and \( S(\mu) = \text{diag}(s_1(\mu), \ldots, s_n(\mu)) \).

(b) Show that a Newton direction can be found by solving the following system of equations.

\[
\begin{align*}
Ad_x^k &= 0, \\
-Qd_x^k + A'p^k + d_s^k &= 0, \\
S_kd_x^k + X_kd_s^k &= \mu^ke - X_kS_ke.
\end{align*}
\]

(c) Develop a primal-dual path following interior point algorithm. Implement the algorithm in Matlab and test it in the following example

\[
Q = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}, \quad c = (2, 1), \quad A = [1 \ 1], \quad b = 1.
\]

Report the result and plot the evolution of the algorithm.

Remark: You might want to consult the book by Bertsimas and Tsitsiklis, “Introduction to Linear Optimization”, where a primal-dual algorithm is given for the linear programming case, and some implementation issues are discussed.

Problem 2: Exercise 4.2.1 of [Ber]

You can skip part (c).