SE/EC/ME 724 Advanced Optimization Theory and Methods

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Lecture 12: Outline

1. Lagrange multipliers: sufficient optimality conditions for problems with equality constraints.
2. Sensitivity.
3. Inequality constraints: Karush-Kuhn-Tucker conditions.

Sufficient optimality conditions

Define the **Lagrangian Function**: $L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x})$.

Proposition

Let $\mathbf{x}^*, \boldsymbol{\lambda}^*$ satisfy

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0, \qquad \nabla_{\boldsymbol{\lambda}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$$

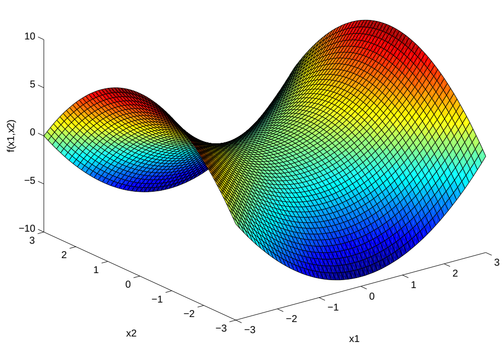
$$\mathbf{y}' \nabla^2_{\mathbf{xx}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) \mathbf{y} > 0, \qquad \forall \mathbf{y} \neq \mathbf{0} \text{ s.t. } \nabla \mathbf{h}(\mathbf{x}^*)' \mathbf{y} = \mathbf{0}.$$

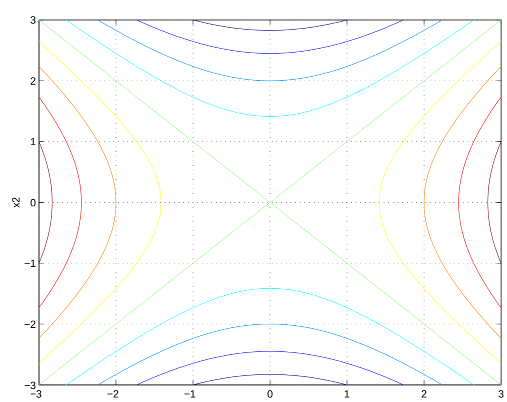
Then $\mathbf{x}^*$ is a **strict** local min of f over $\mathbf{h}(\mathbf{x}) = \mathbf{0}$, i.e., $\exists \gamma, \epsilon > 0$ s.t.

$$f(\mathbf{x}) \geq f(\mathbf{x}^*) + \frac{\gamma}{2} ||\mathbf{x} - \mathbf{x}^*||^2, \qquad \forall \mathbf{x} \text{ s.t. } \mathbf{h}(\mathbf{x}) = \mathbf{0} \text{ with } ||\mathbf{x} - \mathbf{x}^*|| < \epsilon.$$

Example

$$\begin{aligned} \min \quad & f(\mathbf{x}) = x_1^2 - x_2^2 \\ \text{s.t.} \quad & x_1^2 + 2x_2^2 = 4 \end{aligned}$$





Problems with equality and inequality constraints

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})) = \mathbf{0}, \\ & \mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_r(\mathbf{x})) \leq \mathbf{0}, \end{aligned}$$

where f, h_i, g_j are continuously differentiable (in a open set containing the minimum).

Define the **Lagrangian Function**:
$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) + \sum_{j=1}^{r} \mu_j g_j(\mathbf{x})$.

Let $A(\mathbf{x}) = \{j \mid g_j(\mathbf{x}) = 0\}$.

Karush-Kuhn-Tucker necessary optimality conditions

Proposition

Let $\mathbf{x}^*$ be a regular local minimum, i.e., $\nabla h_1(\mathbf{x}^*), \ldots, \nabla h_m(\mathbf{x}^*), \nabla g_j(\mathbf{x}^*)$, $j \in A(\mathbf{x}^*)$ are linearly independent. Then, there exists Lagrange multiplier vectors $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_m^*)$ and $\boldsymbol{\mu}^* = (\mu_1^*, \ldots, \mu_r^*)$ s.t.

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0, \qquad \boldsymbol{\mu}^* \geq \mathbf{0},$$
$$\mu_j^* = 0, \qquad \forall j \notin A(\mathbf{x}^*)$$

If $f, \mathbf{h}, \mathbf{g}$ are twice continuously differentiable

$$\mathbf{y}' \nabla_{\mathbf{xx}}^2 L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{y} \geq 0, \qquad \forall \mathbf{y} \in V(\mathbf{x}^*),$$

where $V(\mathbf{x}^*) = \{\mathbf{y} \mid \nabla h_i(\mathbf{x}^*)' \mathbf{y} = 0,\ \forall i,\ \nabla g_j(\mathbf{x}^*)' \mathbf{y} = 0,\ j \in A(\mathbf{x}^*)\}$.

Sufficient conditions

Proposition

Let $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*$ satisfy

$$\begin{gathered}
\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0, \qquad \mathbf{h}(\mathbf{x}^*) = \mathbf{0}, \qquad \mathbf{g}(\mathbf{x}^*) \le \mathbf{0} \\
\boldsymbol{\mu}^* \ge \mathbf{0}, \qquad \mu_j^* = 0,\ \forall j \notin A(\mathbf{x}^*), \\
\mathbf{y}' \nabla^2_{\mathbf{xx}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{y} > 0, \qquad \forall \mathbf{y} \ne \mathbf{0},\ \mathbf{y} \in V(\mathbf{x}^*), \\
\mu_j^* > 0,\ \forall j \in A(\mathbf{x}^*).
\end{gathered}$$

Then $\mathbf{x}^*$ is a strict local minimum of the constrained problem.