Lecture 10: Outline

- Lagrange multipliers: necessary optimality conditions for problems with equality constraints.
Problems with equality constraints

\[
\min f(x) \\
\text{s.t. } h_i(x) = 0, \quad i = 1, \ldots, m
\]

where \( f, h_i \) are continuously differentiable (in a open set containing the minimum).

Letting \( h = (h_1, \ldots, h_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) we have

\[
\min f(x) \\
\text{s.t. } h(x) = 0
\]

Necessary optimality conditions

**Proposition**

Let \( x^* \) be a local minimum. Assume that \( x^* \) is regular, i.e., \( \nabla h_1(x^*), \ldots, \nabla h_m(x^*) \) are linearly independent. Then, there exists a Lagrange multiplier vector \( \lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \) s.t.

\[
\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) = 0.
\]

If \( f, h \) are twice continuously differentiable

\[
y' \left( \nabla^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(x^*) \right) y \geq 0, \quad \forall y \in V(x^*),
\]

where \( V(x^*) = \{ y \mid \nabla h_i(x^*)'y = 0, \ i = 1, \ldots, m \} \) is the subspace of first order feasible directions.