Lecture 2: Outline

- Background material: subspaces, polyhedra, hulls, norms, continuity, limits.
- Convexity.
Polyhedral Geometry: Some Definitions

**Polyhedron** is a set of the form \( \{ x \in \mathbb{R}^n \mid Ax \geq b \} \).

**Polyhedron in standard form** is a set of the form \( \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \} \).

**Polytope** is a bounded polyhedron (i.e., \( \exists K \) s.t. \( |x_i| \leq K \) \( \forall x = (x_1, \ldots, x_n) \in \text{polyhedron} \)).

**Hyperplane** is a set of the form \( \{ x \in \mathbb{R}^n \mid a'x = b \} \).

**Halfspace** is a set of the form \( \{ x \in \mathbb{R}^n \mid a'x \geq b \} \).

A polyhedron is the intersection of a finite number of halfspaces \( (Ax \geq b \Rightarrow a^i'x \geq b_i, \ i = 1, \ldots, m) \).

\( S \subset \mathbb{R}^n \) is **convex** if for any \( x, y \in S \) and any \( \lambda \in [0, 1] \):

\[ \lambda x + (1 - \lambda)y \in S. \]

Let \( \lambda_i \geq 0 \) \( \sum_{i=1}^{k} \lambda_i = 1 \). The vector \( \sum_{i=1}^{k} \lambda_i x_i \) is said to be **convex combination** of \( x_i \)'s.  

**Convex hull** of \( x_i \)'s is the set of all their convex combinations.

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Properties of Convex Sets

**Theorem**

- The intersection of convex sets is convex.
- Polyhedra are convex sets.
- Convex combination of finite number of elements of a convex set belongs to the set.
- Convex hull of a finite number of vectors is a convex set.
- The vector sum \( \{ x_1 + x_2 \mid x_1 \in C_1, x_2 \in C_2 \} \) of convex sets \( C_1, C_2 \) is convex.
- The image of a convex set under a linear transformation is convex.
Real Analysis background

- Norms $\| \cdot \|$ on $\mathbb{R}^n$.
- Euclidean norm: $\| x \| = \sqrt{x^t x}$.
- Limits.
- Open Ball around $a$ with radius $r$: $\{ y \mid \| y - a \| < r \}$.
- $\mathcal{A} \subset \mathbb{R}^n$ is compact if and only if (iff) closed and bounded (Heine-Borel).
- Consider function $f : \mathcal{A} \to \mathbb{R}^n$:
  - **continuous** at $x \in \mathcal{A}$ if $\lim_{y \to x} f(y) = f(x)$.
  - **right-continuous** if $\lim_{y \uparrow x} f(y) = f(x)$.
  - **left-continuous** if $\lim_{y \downarrow x} f(y) = f(x)$.
  - **lower-semicontinuous** if $f(x) \leq \liminf_{k \to \infty} f(x_k)$ for every sequence $x_k \to x$.
  - **upper-semicontinuous** if $f(x) \geq \limsup_{k \to \infty} f(x_k)$ for every sequence $x_k \to x$.
  - **coercive** if $\lim_{k \to \infty} f(x_k) = \infty$ for every sequence satisfying $\| x_k \| \to \infty$. 