Lecture 22: Outline

- Families of Integer Programming methods.
- Cutting plane methods.
- Branch and bound.
- Dynamic programming: Traveling salesman problem.
- IP duality.
- Approximation algorithms.
- Heuristics.
Integer Programming methods

- **Exact Methods** (EXP-time): cutting plane, branch and bound, DP.
- **Approximation Methods** (POLY-time): suboptimal solutions.
- **Heuristic Methods**: local search, simulated annealing.

### Cutting Plane methods

\[
\begin{align*}
\min & \quad c'x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0 \\
& \quad x : \text{integer}
\end{align*}
\]

**Cutting plane algorithm:**

- Solve LP relaxation of IP \( \Rightarrow x^* \).
- If \( x^* \) integer STOP.
- Otherwise “cut” \( x^* \): add violating constraint which is satisfied by all IP feasible solutions.

How do we cut? **Gomory cuts.**
Branch and Bound

- Solve LP relaxation $\Rightarrow x^*$. Select fractional component $x^*_i$.
- Create two subproblems $F_1, F_2$ by adding either of:
  
  $$x_i \leq \lfloor x^*_i \rfloor \quad \text{or} \quad x_i \geq \lceil x^*_i \rceil$$

  Make both active.
- LP relaxation of each subproblem $F_i$ yields lower bound $b(F_i)$.
- Maintain upper bound $U$ on IP optimal cost ($U$ obtained by evaluating the cost of an IP feasible sol., e.g., when a subproblem has an integer optimal sol.)

Branch and Bound algorithm

- Initialize $U = \infty$.
- Select an active subproblem.
- Consider LP relaxation. If infeasible delete subproblem. Else solve.
- If optimal solution integer delete subproblem and update $U$. Else compute $b(F_i)$.
- If $b(F_i) \geq U$, delete subproblem.
- If $b(F_i) < U$ divide into two further subproblems.
Approximation algorithms

**Definition**

For a minimization problem: An algorithm is an \( \epsilon \)-approximation algorithm if it runs in POLY-time and returns a feasible solution \( Z \) such that

\[
Z \leq (1 + \epsilon)Z^*
\]

For maximization:

\[
Z \geq (1 - \epsilon)Z^*
\]