Lecture 8: Outline

1. General form of the dual.
2. Weak duality.
3. Strong duality.
5. Relations between primal and dual.
Constructing the Dual

Consider the LP with optimal solution $x^*$

$$\textbf{Primal} \quad \min \; c'x$$
$$\text{s.t.} \quad Ax = b$$
$$\quad x \geq 0$$

Relax the constraint by introducing the vector of Lagrange multipliers (or dual variables) $p$

$$g(p) = \min \; c'x + p'(b - Ax)$$
$$\text{s.t.} \quad x \geq 0$$

Note: $g(p) \leq c'x^*$. Get the tightest lower bound

$$\textbf{Dual} \quad \max g(p) \Leftrightarrow \max p'b$$
$$\text{s.t.} \; p'A \leq c'$$

General form of the dual

$$\textbf{Primal} \quad \min \; c'x$$
$$\text{a}_i'x \geq b_i; \quad i \in M_1$$
$$\text{a}_i'x \leq b_i; \quad i \in M_2$$
$$\text{a}_i'x = b_i; \quad i \in M_3$$
$$x_j \geq 0; \quad j \in N_1$$
$$x_j \leq 0; \quad j \in N_2$$
$$x_j \leq 0; \quad j \in N_3$$

$$\textbf{Dual} \quad \max \; p'b$$
$$p_i \geq 0; \quad i \in M_1$$
$$p_i \leq 0; \quad i \in M_2$$
$$p_i \leq 0; \quad i \in M_3$$

$$\text{p}'A_j \leq c_j; \quad j \in N_1$$
$$\text{p}'A_j \geq c_j; \quad j \in N_2$$
$$\text{p}'A_j = c_j; \quad j \in N_3$$

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Properties

**Theorem**
The dual of the dual is the primal.

**Theorem**
*(Weak Duality)* If \( x \) is primal feasible and \( p \) is dual feasible then \( p'b \leq c'x \).

**Corollary**
If \( x \) is primal feasible, \( p \) is dual feasible, and \( p'b = c'x \), then \( x \) is optimal in the primal and \( p \) is optimal in the dual.

**Theorem**
*(Strong Duality)* If LP has optimal solution, then so does the dual, and the optimal costs are equal.

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Relations between primal and dual

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