Formulations Nonlinear Objectives Geometry

Lecture 2: Outline

- Applications of LP and formulations.
- Nonlinear objective functions.
- Linear Algebra Review.
- Geometric concepts: polyhedra and convex sets.
Examples and formulations

**How to formulate?** There is no systematic method ...

- Think to define decision variables.
- Translate objective and problem data to constraints and objective function.
- **Be parsimonious!**

Some formulations:

- The Diet Problem.
- Production Planning in a manufacturing plant.
- Routing in a communication network.
- Maximum lifetime routing in wireless sensor networks.
- Flux balance analysis in metabolic networks.

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**Piecewise linear convex functions**

\[
\min \max_{i=1,\ldots,m} (c_i^T x + d_i) \\
\text{s.t.} \quad A x \geq b \\
\max_{j=1,\ldots,k} (f_j^T x + g_j) \leq h
\]

equivalent to LP

\[
\min z \\
\text{s.t.} \quad z \geq c_i^T x + d_i, \quad i = 1, \ldots, m \\
A x \geq b \\
f_j^T x + g_j \leq h, \quad j = 1, \ldots, k
\]
**Absolute values**

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} c_i |x_i| \quad (c_i \geq 0) \\
\text{s.t.} & \quad Ax \geq b
\end{align*}
\]

is equivalent to LP

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} c_i z_i \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x_i \leq z_i \\
& \quad -x_i \leq z_i
\end{align*}
\]

and also equivalent to LP

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} c_i (x_i^+ + x_i^-) \\
\text{s.t.} & \quad A(x^+ - x^-) \geq b \\
& \quad x^+, x^- \geq 0
\end{align*}
\]

Proof as Exercise ...

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**Polyhedral Geometry: Some Definitions**

**Polyhedron** is a set of the form \( \{ x \in \mathbb{R}^n \mid Ax \geq b \} \).

**Polyhedron in standard form** is a set of the form \( \{ x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0 \} \).

**Polytope** is a bounded polyhedron (i.e., \( \exists K \) s.t. \( |x_i| \leq K \forall x = (x_1, \ldots, x_n) \in \text{polyhedron} \)).

**Hyperplane** is a set of the form \( \{ x \in \mathbb{R}^n \mid a'x = b \} \).

**Halfspace** is a set of the form \( \{ x \in \mathbb{R}^n \mid a'x \geq b \} \).
A polyhedron is the intersection of a finite number of halfspaces $\left( Ax \geq b \Rightarrow a'_i x \geq b_i, \ i = 1, \ldots, m \right)$.

$S \subset \mathbb{R}^n$ is convex if for any $x, y \in S$ and any $\lambda \in [0, 1]$: 
$\lambda x + (1 - \lambda) y \in S$.
Let $\lambda_i \geq 0 \ \sum_{i=1}^{k} \lambda_i = 1$. The vector $\sum_{i=1}^{k} \lambda_i x_i$ is said to be convex combination of $x_i$'s.

Convex hull of $x_i$'s is the set of all their convex combinations.

Properties of Convex Sets

**Theorem**

- The intersection of convex sets is convex.
- Polyhedra are convex sets.
- Convex combination of finite number of elements of a convex set belongs to the set.
- Convex hull of a finite number of vectors is a convex set.