Lecture 10: Outline

- Farkas’ lemma.
- Application: Asset Pricing in an arbitrage-free environment.
- Cones and extreme rays.
- Unboundness conditions.
- Resolution Theorem.
Farkas’ lemma

**Theorem**

(Farkas’ lemma) Exactly one of the following two alternatives hold:

- \( \exists x \geq 0 \text{ s.t. } Ax = b \).
- \( \exists p \text{ s.t. } p'A \geq 0' \text{ and } p'b < 0 \).

**Corollary**

Assume that any \( p \) satisfying \( p'A_i \geq 0 \), also satisfies \( p'b \geq 0 \). Then \( b \) can be written as a nonnegative lin. combination of \( A_1, \ldots, A_n \).

**Theorem**

Suppose \( Ax \leq b \) has at least one feasible solution. Let \( d \) scalar. Then the following are equivalent:

- \( \forall \text{ feasible sols. of } Ax \leq b \text{ we have } c'x \leq d \).
- \( \exists p \geq 0 \text{ s.t. } p'A = c' \text{ and } p'b \leq d \).

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Cones

**Definition**

A set \( C \subset \mathbb{R}^n \) is a **cone** if \( \lambda x \in C \forall \lambda \geq 0 \text{ and } \forall x \in C \).

**Definition**

**Polyhedral cone**: \( \{x \in \mathbb{R}^n \mid Ax \geq 0\} \). If \( 0 \) is an extreme point we have a **pointed polyhedral cone**.

**Theorem**

Let polyhedral cone \( C \subset \mathbb{R}^n \) s.t. \( C = \{x \mid a'_i x \geq 0\} \). Then the following are equivalent:

- \( 0 \) is an extreme point of \( C \).
- \( C \) does not contain a line.
- \( \exists n \text{ lin. ind. vectors in } a'_1, \ldots, a'_m \).
Recession cones and extreme rays

Let nonempty polyhedron \( P = \{ x \mid Ax \geq b \} \).

**Definition**

**Recession cone at** \( y \in P \):

\[
\{ d \mid A(y + \lambda d) \geq b \ \forall \lambda \geq 0 \} \Rightarrow \{ d \mid Ad \geq 0 \}
\]

\( d \) in recession cone are called **rays** of polyhedron.

**Extreme rays of polyhedral cone** \( C \): \( x \in C \) s.t. \( n - 1 \) lin. ind. constraints are active at \( x \).

Extreme rays of recession cone of \( P \) are called **extreme rays of** \( P \).

**Unboundness Conditions**

**Theorem**

Consider \( \min c'x \) over pointed \( C = \{ x \mid a_i'x \geq 0 \} \). Cost = \( -\infty \) iff \( \exists \) extreme ray \( d \) with \( c'd < 0 \).

**Theorem**

Consider \( \min c'x \) over \( Ax \geq b \). Assume at least one extreme point exists. Cost = \( -\infty \) iff \( \exists \) extreme ray \( d \) with \( c'd < 0 \).
Resolution Theorem

**Theorem**

Let \( P = \{ x \in \mathbb{R}^n \mid Ax \geq b \} \neq \emptyset \) with at least one extreme point. Let \( x^1, \ldots, x^k \) be the extreme points and \( w^1, \ldots, w^r \) a complete set of rays. Then

\[
Q \triangleq \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \mid \lambda_i \geq 0, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\} = P
\]

Converse is also true: Every set of the form of \( Q \) (finitely generated) is a polyhedron.