# Algebra for Lemma 6 of "Competition Policy as Strategic Trade"*

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Let $\beta_x = xP''/P'$ and $\beta_y = yP''/P'$. Note that the condition that $QP''/P'$ is constant implies that $\beta_x$ and $\beta_y$ are constant. In this notation, the optimal subsidy is defined by:

$$s_h = P'x\left(\frac{n_h}{n_f(\beta_y+1)+1} - 1\right)$$

Therefore:

$$\frac{ds_h}{dn_f} = P'x\frac{-n_h(\beta_y+1)}{(n_f(\beta_y+1)+1)^2} + \left(\frac{n_h}{n_f(\beta_y+1)+1} - 1\right)\frac{dP'x}{dn_f} \tag{1}$$

$$\begin{aligned}
\frac{dP'x}{dn_f} &= P'\left(\frac{\partial x}{\partial n_f} + \frac{\partial x}{\partial s_h}\frac{ds_h}{dn_f} + \frac{\partial x}{\partial s_f}\frac{ds_f}{dn_f}\right) + xP''\left(\frac{\partial Q}{\partial n_f} + \frac{\partial Q}{\partial s_h}\frac{ds_h}{dn_f} + \frac{\partial Q}{\partial s_f}\frac{ds_f}{dn_f}\right) \\
&= P'\frac{\partial x}{\partial n_f} + xP''\frac{\partial Q}{\partial n_f} + \left(P'\frac{\partial x}{\partial s_h} + xP''\frac{\partial Q}{\partial s_h}\right)\frac{ds_h}{dn_f} + \left(P'\frac{\partial x}{\partial s_f} + xP''\frac{\partial Q}{\partial s_f}\right)\frac{ds_f}{dn_f}
\end{aligned}$$

*Not for publication.

Note that from the lemmas in the paper we have:

$$
\begin{aligned}
\frac{\partial x}{\partial n_f} &= \frac{-y(P' + P''x)}{n_h(P' + P''x) + n_f(P' + P''y) + P'} = \frac{-y(\beta_x + 1)}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1} \\
\frac{\partial Q}{dn_f} &= y + n_h\frac{\partial x}{\partial n_f} + n_f\frac{\partial y}{\partial n_f} = y\left(1 + \frac{-n_h(P' + P''x) - n_f(P' + P''y)}{n_h(P' + P''x) + n_f(P' + P''y) + P'}\right) \\
&= \frac{yP'}{n_h(P' + P''x) + n_f(P' + P''y) + P'} = \frac{y}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1} \\
\frac{\partial x}{\partial s_h} &= \frac{-(n_f(P' + P''y) + P')}{(n_h(P' + P''x) + n_f(P' + P''y) + P')P'} = \frac{-n_f(\beta_y + 1) - 1}{(n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1)P'} \\
\frac{\partial Q}{\partial s_h} &= n_h\frac{\partial x}{\partial s_h} + n_f\frac{\partial y}{\partial s_h} = \frac{-n_h(n_f(P' + P''y) + P') + n_f n_h(P' + P''y)}{(n_h(P' + P''x) + n_f(P' + P''y) + P')P'} \\
&= \frac{-n_h}{n_h(P' + P''x) + n_f(P' + P''y) + P'} = \frac{-n_h}{\left(n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1\right)P'} \\
\frac{\partial x}{\partial s_f} &= \frac{n_f(P' + P''x)}{(n_h(P' + P''x) + n_f(P' + P''y) + P')P'} = \frac{n_f(1 + \beta_x)}{\left(n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1\right)P'} \\
\frac{\partial Q}{\partial s_f} &= n_h\frac{\partial x}{\partial s_f} + n_f\frac{\partial y}{\partial s_f} = \frac{n_h n_f(P' + P''x) - n_f(n_h(P' + P''x) + P')}{(n_h(P' + P''x) + n_f(P' + P''y) + P')P'} \\
&= \frac{-n_f}{n_h(P' + P''x) + n_f(P' + P''y) + P'} = \frac{-n_f}{\left(n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1\right)P'}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
P'\frac{\partial x}{\partial n_f} + xP''\frac{\partial Q}{\partial n_f} &= \frac{-P'y(\beta_x + 1) + P''xy}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1} = \frac{-P'y}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1} \\
P'\frac{\partial x}{\partial s_h} + xP''\frac{\partial Q}{\partial s_h} &= \frac{-P'(n_f(\beta_y + 1) + 1) - P''xn_h}{\left(n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1\right)P'} = -\frac{(n_f + 1) + n_h\beta_x + n_f\beta_y}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1} \\
P'\frac{\partial x}{\partial s_f} + xP''\frac{\partial Q}{\partial s_f} &= \frac{P'n_f(1 + \beta_x) - P''xn_f}{\left(n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1\right)P'} = \frac{n_f}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1}
\end{aligned}
$$

Plugging into Equation 1 and rearranging gives:

$$
\begin{aligned}
&\frac{ds_h}{dn_f}\left(1 + \frac{n_h - n_f(\beta_y + 1) - 1}{n_f(\beta_y + 1) + 1}\,\frac{n_f + 1 + n_h\beta_x + n_f\beta_y}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1}\right) \\
= \; & P'x\frac{-n_h(\beta_y + 1)}{(n_f(\beta_y + 1) + 1)^2} + \frac{n_h - n_f(\beta_y + 1) - 1}{n_f(\beta_y + 1) + 1}\,\frac{-P'y}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1} \\
&+ \frac{n_h - n_f(\beta_y + 1) - 1}{n_f(\beta_y + 1) + 1}\,\frac{n_f}{n_h(\beta_x + 1) + n_f(\beta_y + 1) + 1}\,\frac{ds_f}{dn_f}
\end{aligned}
$$

Now I impose symmetry. That is, $n_h = n_f = n$, $x = y$ and $\beta_x = \beta_y = \beta$:

$$\begin{aligned}
&\frac{ds_h}{dn_f}\left(1+\frac{-n\beta-1}{n(\beta+1)+1}\,\frac{n(2\beta+1)+1}{2n(\beta+1)+1}\right)\\
&= P'x\left(\frac{-n(\beta+1)}{(n(\beta+1)+1)^2}+\frac{-n\beta-1}{n(\beta+1)+1}\,\frac{-1}{2n(\beta+1)+1}\right)+\frac{-n\beta-1}{n(\beta+1)+1}\,\frac{n}{2n(\beta+1)+1}\,\frac{ds_f}{dn_f}
\end{aligned}$$

$$\frac{ds_h}{dn_f}=\left(\frac{-n(\beta+1)(2n(\beta+1)+1)}{\sigma(n(\beta+1)+1)}+\frac{n\beta+1}{\sigma}\right)P'x-\frac{n(n\beta+1)}{\sigma}\,\frac{ds_f}{dn_f}$$

$$\begin{aligned}
\sigma &= (n(\beta+1)+1)(2n(\beta+1)+1)-(n\beta+1)(n(2\beta+1)+1)\\
&= 2n^2\beta^2+2n^2\beta+2n\beta+2n^2\beta+2n^2+2n^2+2n+n\beta+1-[2n^2\beta^2+2n\beta+n^2\beta+n+n\beta+1]\\
&= n(3n\beta+2n+2)
\end{aligned}$$

Note that:

$$\begin{aligned}
&-n(\beta+1)(2n(\beta+1)+1)+(n\beta+1)(n(\beta+1)+1)\\
=\;&-(2n^2\beta^2+2n^2\beta+n\beta+2n^2\beta+2n^2+n)+(n^2\beta^2+n\beta+n^2\beta+n+n\beta+1)\\
=\;&-n^2\beta^2-3n^2\beta-2n^2+n\beta+1
\end{aligned}$$

Therefore:

$$\frac{ds_h}{dn_f}=\left(\frac{-n^2\beta^2-3n^2\beta-2n^2+n\beta+1}{n(3n\beta+2n+2)(n(\beta+1)+1)}\right)P'x-\frac{n\beta+1}{3n\beta+2n+2}\,\frac{ds_f}{dn_f}\qquad(2)$$

Now we do the same for $ds_f/dn_f$; or equivalently, $ds_h/dn_h$

$$\frac{ds_h}{dn_h}=\frac{P'x}{n_f(\beta_y+1)+1}+\left(\frac{n_h}{n_f(\beta_y+1)+1}-1\right)\frac{dP'x}{dn_h}$$

$$\frac{dP'x}{dn_h}=P'\frac{\partial x}{\partial n_h}+xP''\frac{\partial Q}{\partial n_h}+\left(P'\frac{\partial x}{\partial s_h}+xP''\frac{\partial Q}{\partial s_h}\right)\frac{ds_h}{dn_h}+\left(P'\frac{\partial x}{\partial s_f}+xP''\frac{\partial Q}{\partial s_f}\right)\frac{ds_f}{dn_h}$$

$$\begin{aligned}
\frac{\partial x}{\partial n_h} &= \frac{-x(P'+P''x)}{n_h(P'+P''x)+n_f(P'+P''y)+P'}=\frac{-x(\beta_x+1)}{n_h(\beta_x+1)+n_f(\beta_y+1)+1}\\
\frac{\partial Q}{\partial n_h} &= x+n_h\frac{\partial x}{\partial n_h}+n_f\frac{\partial y}{\partial n_h}=\frac{x}{n_h(\beta_x+1)+n_f(\beta_y+1)+1}
\end{aligned}$$

$$P'\frac{\partial x}{\partial n_h}+xP''\frac{\partial Q}{\partial n_h}=\frac{-P'x(\beta_x+1)+P''x^2}{n_h(\beta_x+1)+n_f(\beta_y+1)+1}=\frac{-P'x}{n_h(\beta_x+1)+n_f(\beta_y+1)+1}$$

$$\frac{ds_h}{dn_h}\left(1+\frac{n_h - n_f(\beta_y+1)-1}{n_f(\beta_y+1)+1}\,\frac{n_f+1+n_h\beta_x+n_f\beta_y}{n_h(\beta_x+1)+n_f(\beta_y+1)+1}\right)$$
$$= \frac{P'x}{n_f(\beta_y+1)+1} + \frac{n_h-n_f(\beta_y+1)-1}{n_f(\beta_y+1)+1}\,\frac{-P'x}{n_h(\beta_x+1)+n_f(\beta_y+1)+1}$$
$$+\frac{n_h-n_f(\beta_y+1)-1}{n_f(\beta_y+1)+1}\,\frac{n_f}{n_h(\beta_x+1)+n_f(\beta_y+1)+1}\,\frac{\partial s_f}{\partial n_h}$$

With symmetry:

$$\frac{ds_h}{dn_h}\left(1+\frac{-n\beta-1}{n(\beta+1)+1}\,\frac{n(2\beta+1)+1}{2n(\beta+1)+1}\right)$$
$$= P'x\left(\frac{1}{n(\beta+1)+1}+\frac{-n\beta-1}{n(\beta+1)+1}\,\frac{-1}{2n(\beta+1)+1}\right)+\frac{-n\beta-1}{n(\beta+1)+1}\,\frac{n}{2n(\beta+1)+1}\,\frac{ds_f}{dn_h}$$

$$\frac{ds_h}{dn_h} = P'x\left(\frac{2n(\beta+1)+1}{\sigma}+\frac{n\beta+1}{\sigma}\right) - \frac{n(n\beta+1)}{\sigma}\,\frac{ds_f}{dn_h}$$

$$\frac{ds_h}{dn_h} = P'x\left(\frac{3n\beta+2n+2}{\sigma}\right) - \frac{n(n\beta+1)}{\sigma}\,\frac{ds_f}{dn_h}$$
$$= \frac{P'x}{n} - \frac{n\beta+1}{3n\beta+2n+2}\,\frac{ds_f}{dn_h}$$

By symmetry, we have:

$$\frac{ds_f}{dn_f} = \frac{P'x}{n} - \frac{n\beta+1}{3n\beta+2n+2}\,\frac{ds_h}{dn_f}$$

Substituting into Equation 2,

$$\frac{ds_h}{dn_f} = \left(\frac{-n^2\beta^2-3n^2\beta-2n^2+n\beta+1}{n(3n\beta+2n+2)(n(\beta+1)+1)}\right)P'x - \frac{n\beta+1}{3n\beta+2n+2}\,\frac{Px'}{n}$$
$$+\left(\frac{n\beta+1}{3n\beta+2n+2}\right)^2\frac{ds_h}{dn_f}$$

$$\frac{ds_h}{dn_f}\left(1-\left(\frac{n\beta+1}{3n\beta+2n+2}\right)^2\right) = P'x\left(\frac{-n^2\beta^2-3n^2\beta-2n^2+n\beta+1-[(n\beta+1)(n(\beta+1)+1)]}{n(3n\beta+2n+2)(n(\beta+1)+1)}\right)$$

Note that:

$$-n^2\beta^2-3n^2\beta-2n^2+n\beta+1-[(n\beta+1)(n(\beta+1)+1)]$$
$$= -n^2\beta^2-3n^2\beta-2n^2+n\beta+1-[n^2\beta^2+n^2\beta+n\beta+n\beta+n+1]$$
$$= -2n^2\beta^2-4n^2\beta-2n^2-n\beta-n$$

Therefore:

$$\frac{ds_h}{dn_f} = -P'x\frac{3n\beta + 2n + 2}{(3n\beta + 2n + 2)^2 - (n\beta + 1)^2}\frac{2n^2\beta^2 + 4n^2\beta + 2n^2 + n\beta + n}{n(n(\beta + 1) + 1)}$$

Now it remains to sign this term. The term $-P'x$ is positive. The denominator of the second fraction is always positive. The derivative of the numerator of the second fraction is $4n^2\beta + 4n^2 + n$, which is always positive. Therefore it suffices to check this numerator at the lower limit of $\beta$. Assumption 1 (iii) implies that $\beta >$-1. At $\beta = -1$, the numerator of the second fraction is equal to zero. Therefore, $\beta > -1$ implies that the second fraction is positive. The numerator of the first fraction is greater then zero if

$$\beta > -\frac{2}{3} - \frac{2}{3n}$$

The denominator of the first fraction has two roots:

$$(3n\beta + 2n + 2)^2 - (n\beta + 1)^2 = 0$$

, Solution is : $\left\{\beta = -\frac{1}{4}\frac{2n+3}{n}\right\}, \left\{\beta = -\frac{1}{2}\frac{2n+1}{n}\right\}$

That is, the denominator is negative when:

$$-1 - \frac{1}{2n} < \beta < -\frac{1}{2} - \frac{3}{4n}$$

Note that:

$$-\frac{1}{2} - \frac{3}{4n} > -\frac{2}{3} - \frac{2}{3n}$$

because:

$$\frac{1}{12n} < \frac{1}{6} \text{ always.}$$

Therefore, the following condition implies that the first fraction is positive:

$$\beta = \frac{P''Q}{P'n} > -\frac{1}{2} - \frac{3}{4n}$$

Remember that Assumption 1 (iii) implies that $P''Q/P' > -1$, or that $P''Q/P'n > -1/n$. The following condition

$$\frac{-1}{n} > -\frac{1}{2} - \frac{3}{4n}$$

holds whenever:

$$1 < \frac{n}{2} + \frac{3}{4}$$

which is always true. Therefore, Assumption 1 (iii) implies that the first fraction is positive and therfore $ds_h/dn_f > 0$.