JOURNAL OF COMPARATIVE ECONOMICS 2, 1–11 (1978)

Aggregation for Material Balances

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Received July 1, 1977; revised August 1977

Manove, M., and Weitzman, M. L.—Aggregation for Material Balances

In an input–output system, let final demands and gross outputs be iteratively balanced by successive approximations. The speed of convergence will depend, among other things, on the initial choice of gross outputs. Suppose that, using some aggregation weights, aggregate supply is made equal to aggregate demand in the initial plan. The current paper finds the set of aggregation weights that yields speediest convergence. An economic interpretation of the "optimal" aggregation weights is given, and some examples are calculated. *J. Comp. Econ.*, March 1978, **2**(1), pp. 1–11. Boston University, Boston, Mass., and Massachusetts Institute of Technology, Cambridge, Mass.

Journal of Economic Literature Classification Numbers: 052, 113, 222.

Balancing sectoral production flows is perhaps the most critical short run task facing the managers of a centrally planned economy. In that context, supply and demand are equilibrated by the well-known method of "material balances." On a superficial level, this method consists of drawing up a balance sheet for each important commodity. The balance sheet lists all the planned sources for the commodity with the amounts going to each user. The idea is to adjust the supplies and uses of various commodities so as to get the balance sheets for all commodities into balance simultaneously. A good mathematical rendition of the balancing process is difficult to construct, but several writers have endeavored to represent various aspects of it.[2]

We want to examine a fundamental problem of central planning, in general, and of material balances, in particular. Central planners in large economic organizations are usually operating under severe time and cost constraints with respect to the planning process itself. As a result, it is simply impossible for them to know of and work with specific economic items at the most detailed classification level. If that were done, at least hundreds of thousands of items would have to be planned at the center. To avoid this staggering task, central

[1] Our thanks to Oldrich Kyn and to an anonymous referee who provided unusually detailed and useful comments. Errors and omissions are our own fault. This work was supported by a grant from the National Science Foundation.

[2] See, for example, Levine (1959), Montias (1959), and Manove (1971).

0147–5967/78/0021–0001$02.00/0

planners typically deal only with highly aggregated figures. They plan in terms of tons of steel, instead of getting lost in the thousands of possible types and grades. Aggregation is universally accepted as a natural part of real-life planning and administration.

Any kind of aggregation rule requires two types of specifications. First, the rule must specify a partition of the detailed nomenclature into groups of commodities, each group to be aggregated into a single sector. Second, an aggregation weight must be assigned to each commodity within a group, so that quantities can be made commensurable and can be summed.

Suppose, for example, that it has been decided to aggregate oil and coal into a single sector called fuel. Aggregation weights based on one of a number of characteristics could be used: volume, weight, caloric content, value in terms of production cost, market values, etc.

How good a particular aggregation rule is depends on the purpose of the aggregation. We are interested in aggregation rules which speed up the process of material balances. Unfortunately, finding good rules of aggregation seems to be a highly intractable problem, and we have succeeded only with respect to the simplest case, aggregation to a single sector. Here, the question of partitioning the nomenclature disappears, and the analysis reduces to the problem of choosing good aggregation weights.

A REPRESENTATION OF MATERIAL BALANCES

As a framework for selecting a good aggregation mechanism, we must choose a mathematical representation of the material-balances process. The following model is frequently taken as a paradigm that captures, at a high level of abstraction, many essential features of the process.

Let A be a fully disaggregated input–output matrix appropriate to the given economy. The coefficients of A may or may not be known to the planners. It is desired to produce a final demand vector D. The planners want to know the level of gross outputs X^* that is needed to produce D without waste. In other words, they are seeking a solution X^* to the equation

$$X^* = AX^* + D. \tag{1}$$

The solution to (1) is given by

$$X^* = (I - A)^{-1}D. \tag{2}$$

Suppose that the planners either cannot or do not wish to solve for X^* centrally by calculating it from (2). This might occur for a variety of reasons, including ignorance of the A matrix at a level of product nomenclature sufficiently detailed to be useful.

A commonly accepted mathematical rendition of material balances is an iterative method for finding X^* that does not rely on the center knowing

$(I - A)^{-1}$ or even A itself. It is important to stress that this is only a model, not a complete and accurate description of material balances as employed in practice.

Although the center does not know X^*, using past experience, simple projections, and some analytic techniques, it can name a set of initial "control figures," X_0. Hopefully X_0 is fairly close to X^*, although an exact coincidence would be too much to ask for.

The idealized iterative procedure is described as follows: given that at iteration $t - 1$ the gross-output target is X_{t-1}, at iteration t it is specified to be

$$X_t = AX_{t-1} + D. \tag{3}$$

This has a natural interpretation. The right-hand side is the vector of gross inputs that would be used up in producing a gross output of X_t along with a net final demand of D. Rule (3) sets next round's gross-output target equal to the implicit total demand determined in the previous round.

Conceptually, one could think of iterative balancing taking place in tâtonnement fashion or occurring over real time. Either way, an important feature of (3) is that it allows for a decentralized interpretation. On any round, each sector j is presented with a target output level x_j. Sector j determines its corresponding input needs for commodity i, $a_{ij}x_j$. This is presented to the appropriate supplier, sector i. Sector i in its turn adds up the total "orders" placed with it, $\sum_j a_{ij}x_j + d_i$. This becomes its gross-output target on the next round. Thus "supply" is made equal to lagged "demand." Note that the center plays no role in the above procedure, which does not require explicit knowledge of the A matrix by any agent.

Rule (3) implicitly contains the extremely important assumption that there is a lag of exactly one period from the time any sector receives changes in its supply targets to the time it communicates to other sectors changes in its demand for inputs. This lag in communication is constant over all pairs of sectors. In reality, of course, some sectors may communicate to some of their suppliers faster or more often than in other cases. In the remainder of this paper, we will assume that revisions in demand for inputs are communicated between two different sectors exactly once each period, while revisions in demand are communicated within a single sector a large number of times each period. To represent this assumption, the above notation need not be changed, but the A matrix must be understood to be derived from a table of input–output flows in which the diagonal entries have been zeroed out. In other words, the antecedent flow table must represent only net intersectoral flows and omit all intrasectoral flows.

It is easy to show that the successive approximations algorithm (3) must converge. Substituting from (3) and (1), we have

$$X_t - X^* = AX_{t-1} + D - (AX^* + D).$$

Cancelling terms yields

$$X_t - X^* = A(X_{t-1} - X^*).$$

The above equation has the solution

$$X_t - X^* = A^t (X_0 - X^*). \tag{4}$$

Since

$$\lim_{t \to \infty} A^t = 0$$

for all productive input–output systems, we have that

$$\lim_{t \to \infty} X_t = X^*.$$

CENTRAL PLANNING AND MACROBALANCING

In this simple model, the only role of the central planners is to specify the control figures, X_0. From (4), the convergence speed of X_t to X^* will turn on how quickly the sequence of matrices of the form A^t damps out the vector (X_0 − X^*). Since the coefficients in A are more or less fixed in the short run, speedy covergence must be sought in a good specification of X_0; that is, X_0 should be close to X^*, the solution of (1). One way that the center can calculate good values of X_0 is to explicitly solve Eq. (1), not as it stands (ruled out by constraints on the planning process) but on an aggregate level. In this section, we shall investigate an extreme case of this general method: aggregation to one sector.

We will say that control figures and final demands are in macrobalance when the aggregate value of the control figures is consistent with the aggregate value of final demand.[3] This balancing involves the reduction of each vector of goods to a scalar "macro" value. An aggregation weight is specified for each commodity. The "value" of the final demand vector D is calculated using these weights, and the weights are used to compute a scalar input–output coefficient that is an aggregate version of the input–output technology matrix A. These two numbers are used in Eq. (2) to calculate a financial value of the corresponding gross-output target. In order to find the best aggregation weights for use in the macrobalancing process, we proceed to construct a mathematical model.

Let v be a vector of aggregation weights to be used in planning material balances, and a the aggregate economy-wide input–output coefficient. Suppose that we choose control figures, $\hat{X}_0$, which are macrobalanced, i.e., figures which in the aggregate satisfy (1). This one-dimensional version of Eq. (1) is given by

$$v\hat{X}_0 = av\hat{X}_0 + vD. \tag{5}$$

[3] It should be noted that we are using the terms "aggregation" and "macrobalance" in a somewhat specialized sense which may deviate from standard usage in other contexts.

Thus

$$v\hat{X} = vD/(1-a). \tag{6}$$

We are not really concerned with the origin of the macrobalanced control figures $\hat{X}_0$. As a practical matter, there are a variety of ways to transform any set of unbalanced control figures into macrobalance. For example, when any set of unbalanced control figures X_0 is multiplied by the scalar m, defined by

$$m = vD/(1-a)vX_0, \tag{7}$$

those control figures become macrobalanced. Of course, there are many other ways a set of control figures can be brought into macrobalance.

The idea of balancing an initial plan on a macroeconomic level seems intuitively plausible and economically appealing. Obviously, the effectiveness of the procedure in inducing fast convergence of (3) will generally depend on the choice of the aggregate input–output coefficient a and the aggregation weights v. The main statement of the present paper is that in a well-defined sense there exists a "best" a and v. These are none other than the dominant characteristic root, α, of the A matrix and its associated characteristic row vector w, respectively. In other words, the optimal values of $(1/a) - 1$ and v are the expansion rate and the von Neumann prices for an expanding economy with the given A matrix.

THEOREM. *Let α be the dominant positive characteristic root of a diagonalizable*[4]*, indecomposable productive technology matrix A, and let w be the associated characteristic row vector, so that*

$$\alpha w = wA. \tag{8}$$

Let $\hat{X}_0$ be any vector of control figures satisfying (5) *with $v = w$, $a = \alpha$, and $\tilde{X}_0$ any other vector of control figures that does not satisfy* (5). *Then there exists an integer T such that, element by element,*

$$|\hat{X}_t - X^*| < |\tilde{X}_t - X^*|$$

for all $t \geqslant T$, where $\hat{X}_t$ and $\tilde{X}_t$ are the gross-output targets obtained by applying (3) *to $\hat{X}_0$ and $\tilde{X}_0$, respectively, t times.*

Proof. Define $E_t \equiv X_t - X^*$, $\hat{E}_t \equiv \hat{X}_t - X^*$, and $\tilde{E}_t \equiv \tilde{X}_t - X^*$. We have

$$\begin{aligned} w\hat{E}_0 &= w\hat{X}_0 - wX^* \\ &= (\alpha w\hat{X}_0 + wD) - (wAX^* + wD) \\ &= \alpha w\hat{X}_0 - \alpha wX^* \\ &= \alpha w(\hat{X}_0 - X^*) = \alpha w\hat{E}_0 \end{aligned}$$

[4] Assuming that A is diagonalizable makes the mathematics tractable without really restricting the scope of the theorem. This is because any nondiagonalizable matrix can be made diagonalizable by making arbitrarily small variations in its coefficients.

and it follows that

$$(1-\alpha)\, w\hat{E}_0 = 0.$$

Because α is the dominant characteristic root of a productive technology matrix, $\alpha < 1$, so $1-\alpha \neq 0$. Thus $w\hat{E}_0 = 0$.

On the other hand suppose, tentatively, that $w\tilde{E}_0 = 0$, where $\tilde{X}_0$ corresponding to $\tilde{E}_0$ does not satisfy (5) with $v = w$, $a = \alpha$. Then $w\tilde{X}_0 = w\tilde{X}^*$. By (1) and (8) we have

$$wX^* = wAX^* + wD = \alpha wX^* + wD.$$

Substituting $w\tilde{X}_0$ for wX^* shows that $\tilde{X}_0$ satisfies (5), a contradiction. Thus $w\tilde{E}_0 \neq 0$.

From (4) we have

$$E_t = A^t E_0. \tag{9}$$

Let W be the matrix of row-characteristic vectors of A (see footnote 4) and let Λ be the diagonal matrix of corresponding characteristic roots. We then have $WA = \Lambda W$, so that $A = W^{-1}\Lambda W$. By (9), $E_t = W^{-1}\Lambda^t W E_0$. Letting W_i denote the ith row of W (with $w = W_1$); $\bar{W}_j$, the jth column of W^{-1}; and λ_i, the ith diagonal element of Λ (with $\alpha = \lambda_1$), we can write

$$E_t = \alpha^t (wE_0)\bar{W}_1 + \lambda_2{}^t (W_2 E_o)\bar{W}_2 + \cdots + \lambda_n{}^t (W_n E_0)\bar{W}_n. \tag{10}$$

By making the appropriate substitutions, and factoring out α^t, (10) yields

$$\hat{E}_t = \alpha^t \left[\left(\frac{\lambda_2}{\alpha}\right)^t (W_2\hat{E}_0)\bar{W}_2 + \cdots + \left(\frac{\lambda_n}{\alpha}\right)^t (W_n\hat{E}_0)\bar{W}_n \right]. \tag{11}$$

The first term of (10) is not reflected in (11) because $w\hat{E}_0 = 0$.

Also from (10),

$$\tilde{E}_t = \alpha^t \left[(w\tilde{E}_0)\bar{W}_1 + \left(\frac{\lambda_2}{\alpha}\right)^t (W_2\tilde{E}_0)\bar{W}_2 + \cdots + \left(\frac{\lambda_n}{\alpha}\right)^t (W_n\tilde{E}_0)\bar{W}_n \right]. \tag{12}$$

Note that $\bar{W}_1$ is the column-characteristic vector of A corresponding to the dominant characteristic root. Because A is nonnegative and indecomposable, the generalized Frobenius theorem guarantees that $\bar{W}_1$ is strictly positive. As $w\tilde{E}_0 \neq 0$, the absolute value of the first term in brackets in (12) must be strictly positive as well. From the fact that α has a magnitude greater than any of the λ's, we may now conclude that the absolute value of the bracketed terms in (12) will be strictly larger than the absolute value of the bracketed terms in (11) for sufficiently large t. The theorem follows.

It is evident from Eq. (11) that the characteristic-vector weights cause the term with the slowest damping factor to vanish completely. Therefore, in the limit, the difference in the rate of convergence of the iterative balancing procedure when the optimal aggregation weights are used and the rate with nonoptimal weights will be the difference in the magnitude of the dominant characteristic root and that of the characteristic root (possibly complex) with the second largest magnitude. In some technology matrices, this difference may be larger than in others; unfortunately, it is difficult to relate the difference to any meaningful or obvious property of the matrix.

It is interesting and significant that the optimal aggregation weights, w, for iterative plan balancing are other than ordinary market prices p, which typically approximate embodied factor content. Market prices may be all right for determining the general trade-offs available as long run production alternatives, but they are not necessarily the appropriate ones for measuring the state of overall plan balance for an economy. A dollar's worth of imbalance in machinery may be much more difficult to rectify than a dollar's worth of imbalance in food. Loosely speaking the w aggregation weights measure the relative costs of commodity imbalance in terms of the number of corrective iterations needed to achieve convergence. If w_i is significantly higher than w_j it means that an initial imbalance of a given magnitude in sector i will damp out much more slowly then an imbalance of the same magnitude in sector j. Compared with i, the imbalance in j has much more of a built-in tendency to correct itself.

The fact that the von Neumann dual prices have a semi-practical role to play as optimal aggregation weights is rather surprising. The iterative construction of a consistent plan is an area of economic theory that would appear to have nothing in common with the problem of balanced economic growth at a maximal rate. The reason for this seeming coincidence is brought out in the proof of the main result. Each component of $X_t - X^*$ is a weighted sum of the same geometrically declining terms, with only the weights differing by component. The coefficient of decline of the slowest damping term is α (all the others have magnitude less than α). It is this term that predominates in the limit.

On the other hand, the maximal balanced growth rate associated with A is $g = (1/\alpha) - 1$. The von Neumann prices can be interpreted as (proportional to) the incremental effect on the maximal growth rate g of marginally changing the supply availability of a commodity. The same interpretation, of course, holds with respect to the incremental effect on the dominant damping term $\alpha = 1/(1 + g)$. It seems natural that in a context of achieving plan balance, the correct aggregation weights are measuring the appropriate trade-offs between specific commodity imbalances.

AN APPLICATION TO SOVIET INPUT–OUTPUT DATA

The 1966 Soviet value input–output table, as adjusted by Treml (1973), was aggregated to 19 productive sectors to conform with the nomenclature of the Standardized Input–Output Tables of the ECE Countries published by the United Nations Statistical Commission (1972). For the purpose of deriving optimal aggregation weights for iterative plan balancing, the intrasectoral flows (i.e., the diagonal elements of the first quadrant of the table) were zeroed out (as seen previously).

Table 1 lists the optimal aggregation weights for iterative plan balancing associated with this standardized input–output table; that is, the row-characteristic vector of the A matrix. Because the matrix is in value terms, the aggregation weights are presented per ruble of the respective commodities. In other words, the listed weight for commodity i is w_i/p_i, where w_i is the aggregation weight calculated for the commodity in physical terms, and p_i is approximately the market value of the commodity. The commodities with the largest weights are relatively more important in the iterative plan-balancing process than the embodied factor content would imply, while those with the smallest weights are less important.

In general, one would expect those commodities which are produced in a number of stages in different sectors to cause the worst problems in the balancing process. This is because adjusting the output levels of such commodities causes chains of secondary effects. Naturally, one would expect the optimal aggregation weights of these commodities to be relatively high in order that the financial procedure bring them as closely into balance as possible before the iterative balancing procedure begins. The reader should interpret Table 1 with this in mind.

TABLE 1

OPTIMAL AGGREGATION WEIGHTS

Agriculture	0.358
Fuel	0.935
Mining	0.039
Food processing	0.692
Textiles	0.550
Clothing	0.940
Wood products	0.941
Rubber products	1.278
Chemicals	1.077
Petroleum products	1.128
Mineral products	1.143
Metals	1.643
Transportation machinery and equipment	1.620
Machinery	1.062
Electricity	0.975
Construction	1.522
Commerce	1.370
Transportation and communication	0.631
Other	1.333

Table 2 was calculated from the mean results of 20 Monte Carlo experiments using the Soviet input–output matrix. Initial gross-output targets X were assigned a uniform probability distribution given by:

$$\tfrac{1}{2}\bar{X} \leqslant X \leqslant \tfrac{3}{2}\bar{X}$$

with $\bar{X}$ the actual gross output figures in 1966. Each vector X of initial gross-output targets was macrobalanced by scaling as in (7), and then iterated by material balances as in (3). The sum of the absolute values of plan imbalances for all i, i.e., $\sum_i |(X - AX - D)_i|$, is given in lines 1 and 2 where the total size of imbalances in line 1 with no iterations is indexed by 100. As is evident from the table, a macrobalance with optimal aggregation weights leads to considerably smaller errors than its market-value counterpart after only a few iterations.

TABLE 2

PLAN IMBALANCE REMAINING AFTER EACH ITERATION, ASSUMING INITIAL MACROBALANCE

Aggregation weights for macrobalance	Iteration number							
	0	1	2	3	4	5	6	7
1. Producers prices	100	22	3.5	0.88	0.34	0.12	0.05	0.02
2. Optimal aggregation weights	98	22	2.5	0.35	0.06	0.01	0.00	0.00
3. Line 2 as percentage of 1	98	97	72	40	17	7	4	2

At this point, an important question arises. If the center had the information and facilities necessary to calculate the optimal aggregation weights, could not they also calculate perfectly balanced gross-output targets directly, thus obviating the need for iterative plan balancing? The answer, of course, is yes. However, the center does not have sufficient information to calculate either the vector of balanced gross outputs or the optimal aggregation weights. In Soviet-type economies central planners generally do attempt to approximate balanced gross output vectors (e.g., control figures), but they do not try to approximate optimal aggregation weights. Ultimately the usefulness of approximating the optimal aggregation weights will depend both on the difficulty of making such an approximation, and on the robustness of the optimal solution.

With regard to the former, there is a simple and well-known iterative algorithm that can be used to approximate the major row-characteristic vector of an A matrix. Let t index the iterations. Let π_t be a vector of prices and let s_t be the vector whose elements are the total cost of intermediate goods per unit output of each commodity at prices π_t. Let q be a strictly positive vector of quantity weights. Then the iterative procedure is defined by

$$\pi_{t+1} = \frac{\pi_t q}{s_t q} s_t.$$

TABLE 3

COMPARISON OF OPTIMAL AND APPROXIMATELY OPTIMAL AGGREGATION WEIGHTS

Ratios of absolute sum of imbalance (%)	Iteration number							
	0	1	2	3	4	5	6	7
Optimal weights/producers prices	98	97	72	40	17	7	4	2
Approximate optimal weights/ producers prices	98	97	72	41	20	12	11	10

It is easy to show that π_t must converge to the row-characteristic vector w. In each iteration, new prices are formed by uniformly marking up those portions of the current prices contributed by intermediate goods. The markup used is simply the average markup in all sectors inherent in the prices of the current iteration. Presumably the planners could use one or two iterations of this sort in each production period to maintain a reasonable vector of aggregation weights.

To test the robustness of the optimal aggregation weights, computer experiments were conducted with aggregation weights deviating from the optimal ones.

These "approximate" optimal weights were generated randomly from a uniform distribution over an interval of $\pm 10\%$ around the optimal weights. The results of material-balance iterations after a macrobalance performed with these approximate weights, as compared with those when optimal weights were used, is given in Table 3. The numbers record plan imbalance remaining after each iteration with both optimal and approximate optimal weights, as a percentage of plan imbalance remaining when producers' prices are used for macrobalance.

These results indicate that the optimal aggregation prices are reasonably robust, and that it might be worthwhile to use a good approximation of them for planning purposes.

SUMMARY AND CONCLUSION

Many centrally planned economies use the method of material balances as a way of producing consistent plans. We have modeled this process as follows. First, a preliminary list of output targets, the so-called control figures, are issued. Then, the method of successive approximations is used to adjust the control figures so that supplies of outputs will be brought into balance with demands for inputs and final goods. In our model, the material-balances process is interpreted as an iterative procedure whereby sequences of adjustments are made to the control figures.

We have explored the possibility of performing an initial macroeconomic balance on the control figures for the purpose of speeding up the ensuing material-balance process. Aggregation weights are specified, and the control figures are adjusted (if desired, by a scalar multiple) so that the scalar aggregate value of supply will equal the aggregate value of demand. The aggregation weights chosen for this purpose are exceedingly important. We have proved that in the context of one common model of material balances, it is optimal in the limit to use the row-characteristic vector of the technology matrix as the aggregation weights. In simulation experiments with Soviet input–output data, plan imbalances were dramatically reduced by an initial macroeconomic balance with these aggregation weights. Such initial balancing should not be difficult to perform, and we suggest further research into the practicality of their use.

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