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BARGAINING WITH DEADLINES AND IMPERFECT PLAYER CONTROL

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Anecdotal and experimental evidence suggests that bargaining sessions subject to deadlines often begin with cheap talk and rejected proposals. Agreements, if they are reached at all, tend to be concluded near the deadline. We attempt to capture and explain these phenomena in a strategic bargaining model that incorporates a bargaining deadline, the possibility of strategic delay, and a lack of perfect player control over the timing of offers. Imperfect player control is generated by an exogenous uniformly-distributed random delay in offer transmission. Our model has a symmetric Markov-perfect equilibrium, unique at almost all nodes, in which players adopt strategic delay early in the game, make and reject offers later on, and reach agreements late in the game if at all. In equilibrium players miss the deadline with positive probability. The expected division of the surplus is unique and close to an even split.

KEYWORDS: Bargaining, continuous-time games, deadlines, delay, imperfect control, negotiation, strategic delay.

1. INTRODUCTION

BARGAINING OFTEN OCCURS UNDER THE PRESSURE of a deadline. The deadline may be externally imposed, or one of the parties to the negotiation may have adopted the deadline and made a credible commitment to it. Collective bargaining, contract negotiations, and political and international negotiations are often subject to such deadlines.

A bargaining deadline may be defined as a point in time after which the potential value of an agreement is decreased sharply. The existence of a deadline affects the nature of the bargaining process and its outcome. Roth, Murnighan, and Schoumaker (1988) found what they call a "deadline effect" in four separate experimental studies. The setups of their experiments differed in those aspects that they considered theoretically important. Some of the experiments were based on games of complete information but others were not. Yet the common and most unexpected result was that a great majority of agreements were concluded near the end of the bargaining horizon in all four experiments. Surprisingly, delays in agreements were pervasive even in complete information games. In the authors' words:

> There is a striking concentration of agreements reached in the very last seconds before the deadline. This "deadline effect" appears to be quite robust, in that the distribution of agreements over time appears to be much less sensitive to the experimental manipulations than is the distribution of the terms of agreement.[2]

[1] We are very grateful to Motty Perry for extensive discussions of this research and related issues in bargaining models. We would also like to thank Robert Rosenthal for his generous assistance. We thank Martin Hellwig and two referees for a considerable material contribution to this paper.

[2] Roth et al. provided other quotations and references that document the deadline effect in practice. See also Morley and Stephenson (1977, pp. 74–75) for a report of such phenomena in psychology experiments.

More generally, casual observations and experimental evidence suggest that parties who engage in bargaining under a deadline exhibit a complex behavioral pattern. Not only are delays in agreements very common, but parties may fail to reach an agreement before the deadline. Agreements also tend to be fairly moderate even if they are concluded near the deadline. Furthermore, cheap talk and stalling during the early stages of negotiation are frequently observed. And the negotiation process often incorporates large numbers of rejections and counteroffers.[3]

What brings about this complex pattern of behavior? In a setting of incomplete information, the bargaining parties may want to use the time before the deadline to test one-another and find out what their opponents are willing to settle for. Consequently, the observed behavior may result from information gathering and signaling needs. However, both the descriptive literature and the experimental evidence in Roth et al. suggest that observed outcomes are also consistent with a complete-information framework,[4] and it is this question that we explore here.

In this paper, we construct a complete-information bargaining model whose equilibrium reflects the following stylized facts: initial delay, moderate offers, rejected offers, agreements near the deadline, and some failures in reaching agreement. Our interest, here, is not to make the model highly realistic, but rather to explore the type of abstract framework that is consistent with generally observed bargaining behavior.

Many varieties of bargaining models with deadlines appear in the literature; in fact, any finite horizon bargaining model can be interpreted as such (see Stahl (1972) and Harrington (1986)). But the standard alternating-offer models presuppose either discounting or probabilistic breakdown, and they uniformly yield equilibria with immediate agreements (see Rubinstein (1982), and Binmore, Rubinstein, and Wolinsky (1986), etc.).

To obtain equilibria that conform to the stylized facts, we must add two other features to the finite-horizon bargaining model. The first of these is strategic delay. We will say that an alternating-offer model incorporates strategic delay if a player is permitted to postpone the implementation of his move without losing his turn. The second assumption is imperfect player control over the timing of offers during the bargaining session. An exogenous (identically, independently distributed) random delay is associated with the transmission of each offer. That is, when each offer is made, a random length of time elapses before the other player can respond. The surplus is lost if such a random delay carries negotiations past the deadline. In this paper, the random delay is assumed to be uniformly distributed.

[3] See Craver (1986) and Williams (1983) for a description of the negotiating process in a legal context.

[4] See Craver (1986) for a description of the "information phase" of bargaining (Chapter 6) and the full-information "distributive phase" (Chapter 7). Williams (1983) Chapter 4 also discusses the effect of deadlines on negotiations.

There are several practical interpretations of exogenous random delays. First, the random delay may be attributed to the actual, physical transmission time of an offer. For example, parties may have to bargain through an agent, and it may take time for an agent to relay an offer from one party to another (contacting the party, explaining the offer to him, etc.). Second, when offers have a complex or multi-dimensional structure, time may "be needed simply to adjust one's thinking to new ideas. Only once the new ideas are comfortably absorbed can the negotiation proceed."[5] For example, the respondent may wish to consult a lawyer about the implications of the wording, or about the tax advantages of a payment formula. Of course, the time it takes to understand an offer may be in part a strategic variable, but in our model we decompose this period to an irreducible random delay plus a strategic discretionary waiting time.

It is not difficult to show that strategic delay, alone, can induce equilibria with delayed agreements, even in an extremely simple model. Consider a finite-horizon alternating-offer model without discounting, and suppose that players have the option of strategic delay. Then the player who moves at the last possible time can capture the entire surplus. A player who fails to wait until the last possible time can expect to lose the entire surplus to his opponent for that very reason. Thus, the player with the first move will choose to delay until the end of the game. A small positive discount rate will not affect this decision, but it will render the outcome inefficient, because the equilibrium delay now lowers available surplus. However, if the discount rate is sufficiently big, then the equilibrium strategy is to make an offer at the beginning of the game, and this equilibrium offer will be accepted.[6]

Although this simple model supports a delayed-agreement outcome, it cannot produce equilibrium rejection of offers or the expiration of the deadline without an acceptance. The no-rejection equilibrium of almost all finite-horizon complete information bargaining games seems to follow from the fact that backward induction allows a player to compute exactly what the opponent is willing to settle for. Hence whenever a player intends to make an equilibrium offer, it will be one that is just sufficient to induce an acceptance.

Equilibrium rejection is readily induced, however, by the incorporation of uncertainty into the model. Consider the following simple example. There is a unit surplus to be divided between two players, A and B. Only Player A may make an offer to Player B. If Player B accepts the offer, then they divide the surplus accordingly. If Player B does not, then Player A receives 0 but Player B receives his "outside option." The value of this outside option, unknown to Player A when he makes the offer, is either .25 or .75, each with probability .5. In equilibrium, Player B must accept any offer greater than or equal to the value of his outside option. Therefore, if Player A makes an offer of .75, Player B must accept it. But if Player A makes an offer of .25, then from Player A's

[5] Shoenfield and Schoenfield (1988, p. 171).

[6] We would like to thank Martin Hellwig for pointing this out.

point of view, Player B will reject it with probability .5. (Other offers can never occur in equilibrium.) The first strategy gives Player A a payoff of .25; the second, an expected payoff of $(1 - .25).5 = .375$. So in equilibrium, Player A makes the offer .25, which will be rejected with probability .5.

In this example, uncertainty about an opponent's reaction requires the player to decide between making a small offer with a positive probability of rejection and making a larger offer with guaranteed acceptance. Notice that not all probability distributions of the opponent's reaction lead to the possibility of equilibrium rejection. Here, if the probability of Player B's outside option being .25 decreases to .33 (or less), then in equilibrium, Player A makes a .75 offer, which will always be accepted.

In our bargaining game, with imperfect control, a player faces a tradeoff when choosing between a generous and a parsimonious offer: On the one hand, a generous offer is likely to be accepted even if it arrives very quickly, so that the associated probability of an acceptance is large. Of course the potential payoff to the maker of a generous offer is small. A parsimonious offer, on the other hand, is likely to be rejected unless it arrives near the deadline. Consequently, the probability of an acceptance is small, though the potential payoff to the offer maker is large.

When does a player decide to make a generous or parsimonious offer? When little time is left, a rejection and the ensuing random delay is very likely to lead to the expiration of the deadline without an acceptance. Furthermore, it takes a relatively modest offer to secure an acceptance by the opponent. This suggests that when little time is left, it is optimal to make an offer sufficiently generous that it will be accepted no matter how soon it arrives. Conversely, when the time left in the bargaining session is long enough that the exogenous random delay is unlikely to pass the deadline, a player will find it optimal to send a relatively parsimonious proposal. Thus, with imperfect player control, offers will be made and rejected in equilibrium. Moreover, equilibrium acceptance only will occur sufficiently close to the deadline.

Although imperfect player control has its greatest impact near the end of the game, it turns out that strategic delay is used only when there is so much time left in the game that an offer is sure to arrive before the expiration of the deadline. In this case, if a player were to make an offer, he might lose the initiative and allow his opponent to time his counteroffer optimally. Because of this, adopting strategic delay very early in the game becomes a player's equilibrium move.

We can summarize the equilibrium outcome as follows. Initially, a player delays sending his offer, so as not to lose the ability to make an offer at the optimal time. Once there is a positive probability that an offer may not arrive before the deadline, players will begin sending offers immediately. At first, these offers will not be so large as to guarantee acceptance by the opponent; the offer-maker will risk rejection on the chance that the offer will arrive late, when the opponent is under time pressure. Finally, if there is not much time left before the deadline, a player will make an offer sufficiently generous to be accepted whenever it arrives; otherwise he risks a counteroffer and the probable expiration of the deadline.

The paper is organized as follows. In Section 2, we define the model. In Section 3, we describe a candidate equilibrium and show that it conforms to the major stylized facts of real-world bargaining. In Sections 4 and 5, we look at subgames near the deadline: in Section 4 we show that in these subgames, all subgame-perfect equilibria must be identical to the candidate at almost all nodes;[7] in Section 5 we show that on the same subgames, the candidate is a subgame-perfect equilibrium. Then, in Sections 6, 7, and 8, we show that the candidate is a symmetric Markov-perfect equilibrium for the entire game, unique at almost all nodes. Section 9 concludes.

2. A BARGAINING MODEL

In this section, we describe the bargaining game informally and then present its formal definition. We model time as a continuous variable, because this yields closed-form analytical solutions that provide insight into the bargaining process.

Two risk-neutral players, A and B, bargain over the division of an asset of size 1. The game is played during a finite interval of time bounded from above by the deadline. Bargaining proceeds in an alternating fashion. When it is a player's turn to make an offer, he must choose its size and decide when to send it: immediately or at a later time. Once an offer is sent, an exogenously determined random delay begins, and the opposing player cannot respond until it expires. (For convenience, we interpret the random delay as the transmission time of the offer, so that its expiration coincides with the offer's receipt. But one could just as well assume that the delay includes the time required for the recipient to evaluate the offer.) If a player receives an offer after the deadline, the game is over and both players receive 0. If a player receives an offer at or before the deadline, he may accept or reject it. If he accepts the offer, the game ends and he receives the utility indicated by the size of the offer; his opponent receives the remainder of the surplus. If he rejects the offer, it becomes his turn to make a counteroffer. The horizon of play is sufficiently short so that players do not discount future payoffs. Figure 1 illustrates the game.

Player A moves first at time $T_o < 0$. He chooses an offer p (between 0 and 1) and a time $\hat{t}$ (between T_o and 1) to send it. Player B receives the offer at time $t' = \hat{t} + \delta$, where δ is the realization of a random variable $\tilde{\delta} > 0$.[8] The value of δ is not known to either player until after Player A has made his proposal. Player B cannot move until t'.

If $t' > 1$ (B receives the offer after the deadline), the game ends without an acceptance, and both players receive 0. If $t' \leqslant 1$, then at time t', Player B may

[7] The term "almost all" applies in both its linguistic and mathematical usage. Precise statements of these qualifications of uniqueness are presented at the ends of Sections 4, 6, and 8.

[8] If $\tilde{\delta}$ has bounded support, it's maximum length is normalized to 1, so that any offer made after time 0 has a probability of arriving after the deadline.

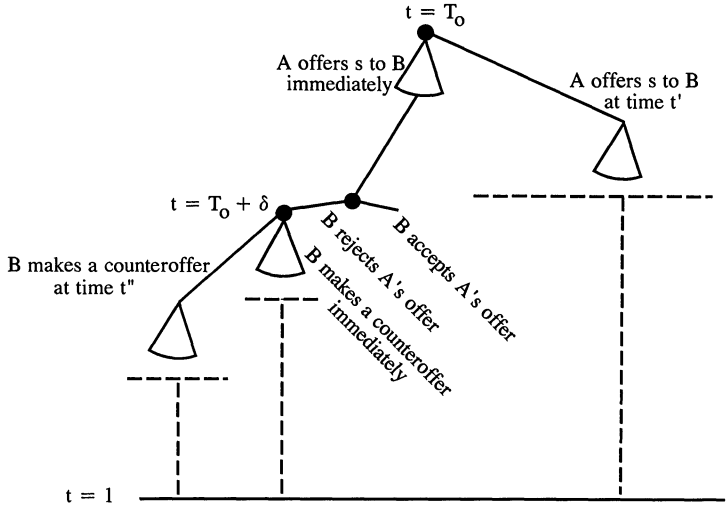

FIGURE 1.—An illustrative game.

accept or reject Player A's offer. If B accepts the offer, he receives utility p and A receives $1-p$. Otherwise, B may choose a counteroffer p' and a time $\hat{t}'$ to send it, where $t' \leqslant \hat{t}' \leqslant 1$. Player A will receive this offer at time $t'' = \hat{t}' + \delta'$, where δ' is another independent realization of $\tilde{\delta}$.

If $t'' > 1$, then the game ends without an acceptance and both players receive 0. Otherwise, Player A faces the same choices at time t'' as Player B did at time t'. The game proceeds in the same alternating fashion until an agreement is reached or until the bargaining deadline is passed, whichever comes first.

Notice that after the first move of the game, each player's turn is divided into two stages. The first stage begins at the time an offer is received. In the first stage, the recipient of the offer may accept or reject it. If the offer is rejected, then the second stage begins immediately. In the second stage, the same player must decide on the size of a counteroffer and a time (current or future) to send it.[9]

The formal definition of game is as follows: The game is played in continuous time over the interval $[T_o, 1]$. Players alternately make and receive offers. An offer is described by the vector $X = (Q, t, p, \hat{t}, \delta)$, where $Q \in \{A, B\}$ designates the player who makes the offer, t is the time when his turn begins, $p \in [0,1]$ is the amount of the offer, $\hat{t} \in [t, 1]$ is the time that the offer is sent, and δ is the realization of the random delay $\tilde{\delta}$ in transmitting the offer.

A valid history of the game, H, is given by $H = (\{X_k\}, Z)$, where $\{X_k\}$, $k = 1, \ldots, \bar{k}$, is a sequence of offers, and where $Z = P$ if the last offer $(\bar{k})$ is pending, and $Z = R$ at the start of the game $(\bar{k} = 1)$ or if the last offer has been

[9] This arrangement is tantamount to adopting a convention that if a player accepts an offer, he accepts it as soon as it is received, an innocuous assumption given a regime of zero time preference.

rejected. The offers must satisfy the following conditions:

a. $Q_1 = A$, $t_1 = T_o$ (when the game opens at time T_o, it is Player A's turn to make an offer);

b. for $k > 1$, $Q_k \neq Q_{k-1}$ (players make offers alternately);

c. for $k > 1$, $t_k = \hat{t}_{k-1} + \delta_{k-1}$ (a player's turn begins when he receives an offer).

These conditions imply that the variables Q_k and t_k are redundant: we use them only for expositional simplicity.

Each valid history constitutes a node of the game, and, in this game, each node defines a subgame. In any subgame, we say that Player Q is facing an outstanding offer if $Q_{\bar{k}} \neq Q$ and $Z = P$, i.e., if the last offer was made by Q's opponent and if it has not been accepted or rejected. We say that Q is making an offer if $Q_{\bar{k}} \neq Q$ and $Z = R$.

Let $\tilde{\mathscr{H}} \equiv \{H\}$ be the set of all valid histories. The set of moves available to Player Q in subgame $H \in \tilde{\mathscr{H}}$ (at time $t = \hat{t}_{\bar{k}} + \delta_{\bar{k}}$) is defined by the mapping M as follows:

$$M(Q,H) = \begin{cases} \{\text{accept, reject}\} & \text{if } Q_{\bar{k}}(t) \neq Q \text{ and } Z = P, \\ \{(p,\hat{t})\} & \text{if } Q_{\bar{k}}(t) \neq Q \text{ and } Z = R, \\ \text{null} & \text{if } Q_{\bar{k}}(t) = Q, \end{cases}$$

where $p \in [0,1]$ represents the amount of a new offer, and $\hat{t} \geqslant t_{\bar{k}}$ denotes the time at which the new offer is to be sent.

A strategy for Player Q is any Borel-measurable function λ that maps the set of valid histories $\tilde{\mathscr{H}}$ into the set of valid moves such that for all $H \in \tilde{\mathscr{H}}$, $\lambda(H) \in M(Q,H)$.

The transition between subgames is defined as follows: When $Z = P$, an acceptance ends the game; a rejection changes the value of Z to R. When $Z = R$, the move $(p,\hat{t})$ implements the new offer $X = (Q,t,p,\hat{t},\delta)$ where $Q \neq Q_{\bar{k}}$, $t = \hat{t}_{\bar{k}} + \delta_{\bar{k}}$, and δ is the new realization of the random delay. If $\hat{t} + \delta > 1$ the game ends without an acceptance; otherwise, X is adjoined to the history of the game, and Z is set to P.

The von Neumann-Morgenstern utility function of each player is defined as follows. If the game ends with an acceptance, then the players' utilities depend on the history H at the end of the game. If $X_{\bar{k}} \equiv (Q_{\bar{k}}, t_{\bar{k}}, p_{\bar{k}}, \hat{t}_{\bar{k}}, \delta_{\bar{k}})$ denotes the last offer in H, the offer maker (Player $Q_{\bar{k}}$), gets utility $1 - p_{\bar{k}}$ and the offer recipient gets utility $p_{\bar{k}}$. Both players get zero if the play of game is without an acceptance.[10] For any pair of Borel-measurable strategies, the expected utility of each player is defined at every node.[11]

[10] Normally, a game is completed without an acceptance because the deadline has expired. But in principle, this outcome also can occur when the infinite sequence of realized δ's sums to less than 1 (a zero-probability event), and the game does not end in a finite number of moves.

[11] It can be demonstrated that for a fixed pair of measurable strategies, utility can be expressed as a measurable function of the histories and sequences of delay realization. This implies that expected utility is a measurable function as well.

Our primary goal here is to characterize a *symmetric Markov-perfect equilibrium* (SMPE) of this game and show it to be unique at almost all nodes.[12] In the context of most bargaining games, Markov-perfect equilibria are composed of strategies that depend only on the current time and value of a pending offer, not on the entire history of play. In particular, these strategies ignore information on all offers that have been rejected. In this game, the relevant information is contained in the state variable that takes the form $\langle Q, p, t\rangle$ when Player Q is facing an outstanding offer p at time t, and the form $\langle Q, R, t\rangle$ at the start of the game or after a rejection at time t, when Q must choose an offer to make to his opponent. We use the expression $\langle Q, x, t\rangle$, where $x \in [0,1] \cup \{R\}$, to encompass both of these cases. And, to economize on notation, we use the same expressions, $\langle Q, p, t\rangle$, $\langle Q, R, t\rangle$, and $\langle Q, x, t\rangle$, to denote classes of subgames with the indicated states.

We say that $\langle Q, x, t\rangle$ is the payoff-relevant state, because payoffs can be represented as a function of $\langle Q, x, t\rangle$, subsequent moves, and subsequent realizations of $\tilde{\delta}$ (see Maskin and Tirole (1988, 1989)). Note that the value of $\langle Q, x, t\rangle$ is determined by part of the history H; namely, $X_{\bar{k}}$, the last offer made, and the value of Z. We will be searching for equilibrium strategies that depend only on $\langle Q, x, t\rangle$.[13]

In the analysis that follows, we assume that the random delay, $\tilde{\delta}$, follows a uniform distribution on the unit interval $[0,1]$. This assumption allows us to solve the game in closed form. We shall discuss other distributions later.

3. A DESCRIPTION OF THE SYMMETRIC MARKOV-PERFECT EQUILIBRIUM

In sections to follow we solve for an SMPE and show it is unique. For now, we describe that SMPE on a more formal level. To do this, we let the function $u(t)$ represent the expected utility a player obtains from making an equilibrium offer at time $t \geqslant 0$. This function, illustrated in Figure 2, will be shown to be unique. We shall also use the constant $\bar{t}$ given by the unique real root of equation $(1-t)^2(2-t) = 1$; numerically, $\bar{t} \doteq .2451$.

[12] In order to investigate the possibility that the SMPE is unique as a subgame-perfect equilibrium, we wrote a numerical algorithm to compute the unique subgame-perfect equilibrium of discrete-time versions of this game. In the discrete-time model, bargaining continues for T periods, and the random delay is uniformly distributed over the first D periods after an offer is made. When D and T are large, we find that the unique subgame-perfect equilibrium corresponds very closely to the symmetric Markov-perfect equilibrium that we find for the continuous-time game with uniform delay distributions. This suggests that the continuous-time game may have a unique subgame-perfect equilibrium, but we have not been able to prove this conjecture for the game as a whole.

[13] On a more formal level, a Markov-perfect equilibrium is defined as follows: Strategies that are functions only of the payoff-relevant state are called Markov strategies. A Markov perfect equilibrium is a pair of Markov strategies that are best responses to each other in every subgame within the class of all Markov strategies. Maskin and Tirole (1988, 1989) and others have shown that a Markov-perfect equilibrium is also subgame-perfect.

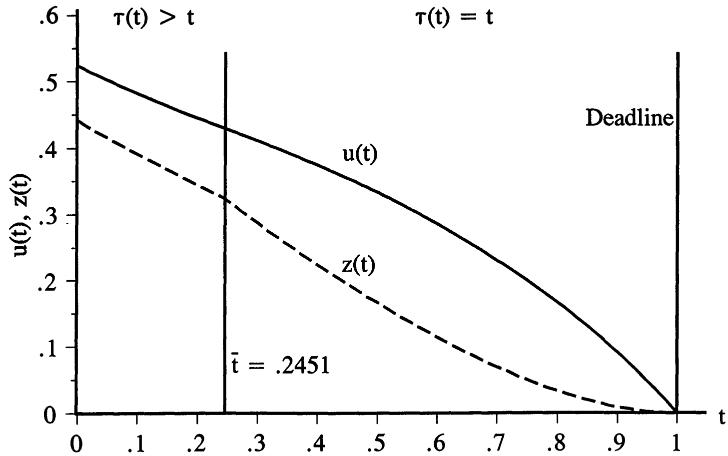

FIGURE 2.—The functions $u(t)$ and $z(t)$ in equilibrium.

The SMPE strategy, denoted in this paper by σ, may now be defined as follows:

> At any time $t \in [T_o, 0)$, a player accepts an offer if and only if the offer is $u(0)$ or higher. At the start of play, or after he rejects an offer at any $t \in [T_o, 0)$, a player will set the size of his offer at $u(\bar{t})$ and send it at time $t = 0$. At any time $t \in [0, 1]$, a player accepts an offer if and only if the offer is $u(t)$ or higher. If a player rejects an offer at time $t \in [0, \bar{t})$, he makes an immediate counteroffer equal to $u(\bar{t})$. If a player rejects an offer at time $t \in [\bar{t}, 1)$, he makes an immediate counteroffer to $u(t)$.

The equilibrium play of the game is as follows. Player A moves first. He waits from time T_o to time 0 and then sends the offer $u(\bar{t}) \doteq .4302$ to Player B. If B receives that offer before $\bar{t} \doteq .2451$, he will reject it and make an immediate counteroffer of the same amount. If A receives B's offer before $\bar{t}$, he also will reject it and make the same counteroffer. If either A or B receives an offer after $\bar{t}$, he will accept it. The first player to receive an offer after $\bar{t}$ will get about 43 percent of the surplus; the player who has made that offer will get about 57 percent.

In equilibrium, no offer will ever be sent before time 0, and offers are accepted only at times sufficiently close to the deadline ($t > \bar{t}$). Any equilibrium offer that arrives after time $\bar{t}$ will be accepted. If bargaining results in an acceptance in equilibrium, the division of the surplus is unique and moderate. However, some equilibrium outcomes have no acceptance. For example, this happens if Player A's offer, sent at time 0, arrives at some time $t' < \bar{t}$, and B's counteroffer, sent at time t', arrives past the deadline because the realized random delay is large. We later show that the expected utility of Player A (the first to move) at time $T_o \leqslant 0$ is $u(0) \doteq .524$ and that of B is approximately .441. This means that the total expected utility of both players is about .965. If an acceptance is obtained, payoffs will sum to 1, so that the *ex-ante* probability of no acceptance must be about 3.5 percent. The *ex-ante* distribution of time of acceptance is uniform on $[\bar{t}, 1]$. Notice that if for any reason the game involves an offer rejected by a player (say A) at a time $t \geqslant \bar{t}$, then the continuation equilibrium will have Player A making an immediate counteroffer equal to $u(t)$. If that counteroffer arrives before the deadline, then Player B's equilibrium response is to accept it.

It will turn out that the equilibrium best response in subgames that begin before $\bar{t}$ differs qualitatively from the best response in subgames that begin at or after $\bar{t}$. Before $\bar{t}$, the best offer is an interior solution with respect to the tradeoff between the size of an offer and its probability of acceptance. At $\bar{t}$ or afterwards, the best offer is a corner solution: as in the standard alternating-offer game, an equilibrium offer is always accepted (if it arrives before the deadline).

Indeed, because of this special property of the equilibrium for subgames beginning at or after $\bar{t}$, we can construct it using a technique similar to that used by Shaked and Sutton (1984) for the Rubinstein (1982) model.[14] Nevertheless, the standard alternating-offer model with probabilistic breakdown (or discounting) is rather different from our own. In that model, probabilistic breakdown may occur only after an offer is rejected and, hence, a player making an offer has exactly the same information as the opponent will have when he decides on a response. Therefore, a player making an offer can predict the opponent's reaction perfectly.

In our model by contrast, a player, say A, faces two kinds of uncertainty when making an offer. Because of the random delay, the offer may or may not arrive before the deadline. More important though, is the fact that the precise arrival time is uncertain. Because of this, A must consider B's entire range of possible responses (acceptance or rejection) contingent on all possible arrival times. This consideration is unnecessary in the standard model. In our model, this uncertainty does not affect the equilibrium outcome for subgames at $\bar{t}$ or afterwards, because in those subgames (and only in those) a player optimally chooses to avoid entirely the risk of the offer being rejected.[15] This may explain why the standard technique works for such subgames, but not others.

4. EQUILIBRIUM ON THE INTERVAL $[\bar{t}, 1]$: UNIQUENESS

The constant $\bar{t}$ is defined as the unique real root of the equation $(1-t)^2(2-t) = 1$ and is approximately equal to .2451. For subgames beginning at any time $t \in [\bar{t}, 1]$ we have obtained a stronger result than for the game in general. In these subgames, the symmetric Markov-perfect equilibrium (SMPE) described in Section 3 takes the form of an almost unique subgame-perfect equilibrium (SPE).[16] In other words, we show that the strategy pair (σ, σ) forms a unique SPE on the interval $[\bar{t}, 1]$. In this section, we show that if any strategy pair

[14] In particular, at time $t \geqslant \bar{t}$, the expression for the equilibrium utility function seems to be identical to a standard model with $1-t$ as the probability of breakdown.

[15] We thank Martin Hellwig for drawing our attention to this interpretation.

[16] We would like to thank a referee for suggesting both this possibility and the general structure of its proof.

(λ_A, λ_B) forms an SPE on $[\bar{t}, 1]$, then, when restricted to $[\bar{t}, 1]$, $\lambda_A = \lambda_B = \sigma$, at almost all nodes.

Consider any subgame described by the state variable $\langle Q, R, t \rangle$ (i.e., any subgame beginning at time t in which it is Player Q's turn to make an offer), and assume that the two players are following a pair of SPE strategies (λ_A, λ_B). We first define two sequences of functions, $\{V_k(t)\}$ and $\{W_k(t)\}$, that converge to the same limit as k increases. We then show that for each k, $V_k(t)$ is an upper bound for Q's equilibrium expected utility (where Q denotes either A or B). Similarly, we show that $W_k(t)$ is a lower bound for Q's equilibrium expected utility. This implies that at Q's turn to make an offer, his equilibrium expected utility is a unique function of time. We follow this with a proof that this particular utility function can be supported by only those strategies that equal σ almost everywhere. In Section 5, we show that (σ, σ) forms an SPE.

The sequences of upper and lower bounds are specified as follows: We define $V_k(t)$ by

$$(1) \qquad V_k(t) \equiv (1-t) - (1-t)^2 + (1-t)^3 - \cdots + (1-t)^{2k+1} \equiv \frac{1-t}{2-t}\left(1 + (1-t)^{2k+1}\right),$$

and we define $W_k(t)$ by

$$(2) \qquad W_k(t) \equiv (1-t) - (1-t)^2 + (1-t)^3 - \cdots + (1-t)^{2k+1} - (1-t)^{2k+2} \equiv \frac{1-t}{2-t}\left(1 - (1-t)^{2k+2}\right).$$

It is straightforward to verify that $V_k(t)$ is strictly decreasing in t and that $W_k(t)$ is strictly concave.

For expositional purposes, we now analyze subgames in the class $\langle A, R, t \rangle$. But all of the propositions and mathematical statements in the remainder of this section are equally valid for subgames in $\langle B, R, t \rangle$ with the roles of A and B reversed.

The proof that $V_k(t)$ and $W_k(t)$ are upper and lower bounds for A's expected utility at time t is by induction. First, $V_o(t) = 1 - t$ is an upper bound for A's expected utility in $\langle A, R, t \rangle$, because this is the probability that any offer made at time t is received before the deadline. We now demonstrate the following proposition.

PROPOSITION 4.1: *If for all $s \in [\bar{t}, 1]$, $V_k(s)$ is an upper bound for A's and B's equilibrium expected utility in subgames in $\langle A, R, s \rangle$ and $\langle B, R, s \rangle$, then $W_k(t)$ is a lower bound for A's equilibrium expected utility in $\langle A, R, t \rangle$.*

PROOF: $V_k(t)$ is monotonically decreasing on $[0, 1]$. If A offers $V_k(t)$ immediately and if the offer arrives before the deadline (probability $= 1 - t$), then B

must accept it, because $V_k(t)$ exceeds the maximum equilibrium utility he can obtain with a counteroffer. Thus, in any SPE, A's equilibrium expected utility must be at least $(1-t)(1-V_k(t)) \equiv W_k(t)$. *Q.E.D.*

We now divide all of A's feasible offers into two intervals, those less than $W_k(t)$ and those greater than or equal to $W_k(t)$. For each case, we show that $V_{k+1}(t)$ is an upper bound for A's expected utility.

PROPOSITION 4.2: *Suppose in subgames in $\langle A, R, t\rangle$, A offers $p < W_k(t)$ to B. Let time $s \in [t, 1]$ be defined implicitly by $W_k(s) = p$ (the concavity of W_k implies that s is uniquely specified). If p arrives at time $r > s$, then A's expected utility will be greater if B accepts p than if B rejects it.*

PROOF: If B rejects p, B will then offer at most $V_k(r)$. (B will not offer more, because A must accept $V_k(r)$ if it arrives before the deadline.) Thus A gets at most $(1-r)V_k(r) < (1-s)V_k(s)$. If B accepts p, then A gets $1-p = 1-W_k(s)$. Now because $(1-s)(1-V_k(s)) \equiv W_k(s)$, we have $(1-s)V_k(s) = 1-s-W_k(s) < 1-W_k(s)$. *Q.E.D.*

PROPOSITION 4.3: *Suppose in a subgame in $\langle A, R, t\rangle$, A offers $p < W_k(t)$ to B. Then A's expected utility (EU) is less than or equal to $V_{k+1}(t)$.*

PROOF: As before, let $s > t$ be defined implicitly by $W_k(s) = p$. Given $p < W_k(t)$, the concavity of W_k implies that the function is above p on the interval $[t, s)$. Therefore, in the case that offer p arrives at time $r < s$, B will reject it, because he can get at least $W_k(r)$. To calculate an upper bound on A's utility for this case, we assume that B chooses his maximum counteroffer, $V_k(r)$, and sends it immediately, and we note that A must accept that counteroffer if it arrives before the deadline (probability $1-r$).

In the case that offer p arrives at time $r \geqslant s$, B may either reject or accept it, but by Proposition 4.2, we obtain an upper bound on A's expected utility by assuming acceptance. Consequently, we have

$$EU \leqslant \int_t^s (1-r)V_k(r)\,dr + (1-s)(1-W_k(s)). \tag{3}$$

Employing (1) and (2), we can show that the derivative of the right-hand side of (3) with respect to s is negative for all s with $\bar{t} \leqslant t < s < 1$. Therefore, we have $EU \leqslant (1-t)(1-W_k(t)) \equiv V_{k+1}(t)$. *Q.E.D.*

We proceed to prove the same result for the case of $p \geqslant W_k(t)$.

PROPOSITION 4.4: *Suppose in a subgame in $\langle A, R, t\rangle$, A offers $p \geqslant W_k(t)$ to B. Then A's expected utility (EU) is less than or equal to $V_{k+1}(t)$.*

PROOF: Suppose A sends the offer p at time $\hat{t} \geqslant t$. If the offer arrives at time $r < 1$, B will either accept it or make a counteroffer of at most $V_k(r)$ (B will not offer more, because $V_k(r)$ must be accepted if it arrives before the deadline). Therefore we have

$$(4) \qquad EU \leqslant \int_{\hat{t}}^{1} \max\{1-p,(1-r)V_k(r)\}\,dr$$

$$\leqslant \int_{t}^{1} \max\{1-p,(1-t)V_k(t)\}\,dr$$

$$= \max\left\{(1-p)(1-t),(1-t)^2 V_k(t)\right\}.$$

Now, $p \geqslant V_k(t)$ by construction, so that

$$(1-p)(1-t) \leqslant (1-W_k(t))(1-t) = V_{k+1}(t),$$

and

$$(1-t)^2 V_k(t) = V_{k+1}(t) - \left[(1-t)-(1-t)^2\right] < V_{k+1}(t).$$

The conclusion follows. *Q.E.D.*

Proposition 4.5: *In subgame-perfect equilibria in subgames in* $\langle Q,R,t\rangle$, $t \in [\bar{t},1]$, *Player* Q*'s expected utility is uniquely specified by the function*

$$(5) \qquad u(t) = \frac{1-t}{2-t}.$$

Proof: Above, we demonstrated that for all k, $V_k(t)$ and $W_k(t)$ are upper and lower bounds for equilibrium expected utility in any SPE. We know from (1) and (2) that both of these sequences converge to $(1-t)/(2-t)$ as k increases. *Q.E.D.*

Note that when A makes an equilibrium offer of $u(t)$ at time t, B has an expected utility given by $z(t) \equiv (1-t)u(t)$ or

$$(6) \qquad z(t) = \frac{(1-t)^2}{2-t}.$$

It remains to characterize the strategies that support the unique utility function $u(t)$.

Proposition 4.6: *In subgames in* $\langle Q,R,t\rangle$ *with* $t \in [\bar{t},1]$, *an SPE strategy must have the following properties: if an offer p received at time t is greater than* $u(t)$, *it is accepted; if p is less than* $u(t)$, *it is rejected; and upon rejecting an offer, a player makes an immediate counteroffer* $u(t)$ *to his opponent.*

Proof: We know that Players A and B have equilibrium utility functions $u(t)$ in subgames in $\langle A,R,t\rangle$ and $\langle B,R,t\rangle$, respectively. Consequently, A would accept any offer greater than $u(t)$, and reject any offer less than $u(t)$. We need prove only that upon rejecting an offer, A would make an immediate counteroffer of $u(t)$.

(a) Player A can obtain $u(t)$ by offering $p = u(t)$ immediately: If the offer arrives before the deadline (probability $1-t$), then B will accept it, and A will receive $1-u(t)$. Therefore, A's equilibrium expected utility must be at least $(1-t)(1-u(t)) \equiv u(t)$.

(b) Player A would not offer $p > u(t)$ to B: For B would surely accept such an offer it it arrives before the deadline, but it generates less utility than $u(t)$ for A.

(c) A would not offer $p < u(t)$ to B: Suppose A does so. Define the time τ by $p = u(\tau)$. We know $\tau > t$. If the offer p arrives at $r < \tau$, it will be rejected. Since total equilibrium utility for both players in subgames in $\langle B, R, r\rangle$ is at most $1-r$, and since B gets $u(r)$ in any SPE, A's equilibrium expected utility after B's rejection must be at most $1-r-u(r)$. If the offer p arrives at time $r > \tau$ it will be accepted. Let EU denote A's expected utility from making the offer p at time $\hat{t} \geqslant t$. We have

$$EU \leqslant \int_{\hat{t}}^{\tau} [(1-r) - u(r)]\, dr + (1-\tau)(1-u(\tau))$$

$$\leqslant \int_{t}^{\tau} \left[\frac{(1-r)^2}{2-r}\right] dr + u(\tau) < u(t),$$

with the last inequality following from the fact that the left-hand side has a negative derivative with respect to τ.

We conclude that in any SPE for subgames in $\langle A, R, t\rangle$, A immediately offers $p = u(t)$ to B. *Q.E.D.*

Proposition 4.6 narrows the class of potential equilibrium strategies to those that are identical to σ everywhere except in subgames in $\langle Q, p, t\rangle$ for which $p = u(t)$. Note that as a class, the exceptional subgames have a zero probability of being reached *no matter what strategies the players follow*.

5. EQUILIBRIUM ON THE INTERVAL $[\bar{t}, 1]$: EXISTENCE

We now show that the strategy pair (σ, σ) forms an SPE for any subgame that begins in the interval $[\bar{t}, 1]$. That is, we show that σ is a best response against itself in all such subgames.

The strategy of the demonstration is as follows: We choose a representative subgame that begins at time $t \geqslant \bar{t}$, in which it is A's turn to make an offer. We show that by using σ in this subgame, A can obtain $u(t)$. Then we show that every allowable strategy yields a utility no greater than $u(t)$.

PROPOSITION 5.1: *Suppose B uses σ. Then A obtains expected utility $u(t)$ if he uses σ in subgames in $\langle A, R, t\rangle$ with $t \geqslant \bar{t}$.*

PROOF: When A uses σ in $\langle A,R,t\rangle$, he makes an immediate offer of $u(t)$. σ will require B to accept the offer $u(t)$ if it is received before the deadline (which happens with probability $(1-t)$), because $u(t) \geqslant u(s)$ for any $s \in [t,1]$. Hence A's expected utility under strategy σ is $(1-t)(1-u(t)) = u(t)$, from the definition of u. *Q.E.D.*

PROPOSITION 5.2: *Let λ be a strategy defined on $\langle A,R,t\rangle$, and let $U(\lambda)$ denote A's expected utility from using λ when B uses σ. Then if $t \geqslant \bar{t}$, we have that $U(\lambda) \leqslant u(t)$.*

PROOF: Suppose in $\langle A,R,t\rangle$, λ requires A to offer $p \geqslant u(t)$ at time $\hat{t}$. If the offer arrives before the deadline (probability $= 1-\hat{t}$), σ will require B to accept it, and A will receive utility $1-p$. Therefore $U(\lambda) = (1-\hat{t})(1-p) \leqslant (1-t)(1-u(t)) = u(t)$.

So suppose that λ requires A to offer $p < u(t)$. Let the set T be defined by

$$T = \{t \in [\bar{t},1] \mid \text{for some } \lambda \text{ on } \langle A,R,t\rangle, U(\lambda) > u(t)\}.$$

If T is empty, the proof is complete, so we assume T is not empty. First we show that $\sup T > \bar{t}$.[17] Choose $t \in [\bar{t},1]$, and choose λ on $\langle A,R,t\rangle$ such that $U(\lambda) > u(t)$. Select $t' > t$ and let λ' represent λ restricted to $\langle A,R,t'\rangle$. Because B's response to an offer is independent of the time that the offer was made, it must be true that $U(\lambda) = U(\lambda')$ conditional on the offer made in $\langle A,R,t\rangle$ arriving at or after t'. Therefore, by setting t' sufficiently close to t, we can bring $U(\lambda')$ arbitrarily close to $U(\lambda) > u(t)$, in which case we have $t' \in T$. Consequently, $\sup T \geqslant t' > t \geqslant \bar{t}$.

Let $\tilde{t}$ denote $\sup T$, and let A follow a strategy λ that requires him to offer $p = u(\tau) < u(t)$ in $\langle A,R,t\rangle$ with $t \in [\bar{t},\tilde{t})$. We compute an upper bound for $U(\lambda)$. There are two cases: $\tau \leqslant \tilde{t}$ and $\tau > \tilde{t}$. We shall only consider the former; the latter is similar.

If B receives A's offer after time τ, he will accept it. If B receives the offer at time $s < \tau$, he will reject it and make a counteroffer $u(s)$. Let the arrival time of B's counteroffer be $r (\geqslant s)$. If $r > \tilde{t}$, A can do no better than to accept $u(s)$, because we know from the definition of $\tilde{t}$ that A obtains at most $u(r) < u(s)$ in $\langle A,R,r\rangle$. If $r \leqslant \tilde{t}$, we use 1 (the total surplus) as an upper bound to A's expected utility. Hence,

$$U(\lambda) \leqslant \int_t^{\tau}\left[\int_s^{\tilde{t}} dr + \int_{\tilde{t}}^{1} u(s)\,dr\right] ds + (1-\tau)(1-u(\tau)). \tag{7}$$

Let $G(\tau,t)$ denote the difference between the right-hand side of (7) and $u(t)$, so that $G(\tau,t) \geqslant U(\lambda) - u(t)$. Note that $G(t,t) = 0$. Below we show that $\partial G/\partial\tau$ is negative for all τ sufficiently close to $\tilde{t}$, and it follows that $G(\tau,t)$ is negative for all t sufficiently close to $\tilde{t}$ and all $\tau > t$. Thus, for all λ in subgames in $\langle A,R,t\rangle$

[17] We would like to thank a referee for suggesting this line of argument.

with t sufficiently close to $\bar{t}$, we have $U(\lambda) < u(t)$. But this contradicts the definition of $\bar{t}$. We conclude that T is empty.

It remains to find $\partial G/\partial\tau$. Differentiating G with respect to τ, and manipulating the results, we obtain

$$(8) \qquad \frac{\partial G}{\partial \tau} = (\bar{t} - \tau)(1 - u(\tau)) + \frac{(1-\tau)^2(2-\tau) - 1}{(2-\tau)^2}.$$

The second term of (8) is negative for any $\tau > \bar{t}$, so that $\partial G/\partial\tau$ will be negative for $\bar{t} - \tau$ sufficiently small. *Q.E.D.*

Proposition 5.1 and 5.2 immediately imply the following proposition.

PROPOSITION 5.3: *The strategy pair* (σ, σ) *forms an SPE for all subgames in* $\langle Q, R, t\rangle$ *with* $t \geqslant \bar{t}$.

Because σ depends only on the state variable $\langle Q, x, t\rangle, (\sigma, \sigma)$ is an SMPE.

6. A SYMMETRIC MARKOV-PERFECT EQUILIBRIUM ON THE INTERVAL [0,1]: UNIQUENESS

In this section, we deal only with Markov strategies (i.e., strategies that depend only on the state variable $\langle Q, x, t\rangle$). Because all subgames with the same state are equivalent with respect to Markov strategies, it will be convenient to think of $\langle Q, x, t\rangle$ as denoting a single node or subgame.

Suppose that there exists an SMPE strategy pair, (λ, λ), for subgames $\langle Q, x, t\rangle$ with $0 \leqslant t \leqslant 1$. From Section 4, we know that when restricted to subgames beginning on $[\bar{t}, 1]$, $\lambda = \sigma$ at almost all nodes.

We define the set Ω of Markov strategies for Player A, that constitute all possible first-round deviations from λ in $\langle A, R, t\rangle$. That is, when restricted to Ω, A may make any offer (at or after time t) in subgame $\langle A, R, t\rangle$, but he must use λ in his subsequent turns. Clearly Ω is a proper subset of all Markov strategies that A can use in subgame $\langle A, R, t\rangle$. Note that λ itself (when restricted to $\langle A, R, t\rangle$) belongs to Ω. Therefore, it is necessary that λ be A's best response within Ω against λ played by B. We use this fact to characterize λ.

To have consistent notation, we let $u(t)$ and $z(t)$ represent the respective expected utilities of players A and B in subgame $\langle A, R, t\rangle$, when both A and B use λ. By symmetry, $u(t)$ and $z(t)$ also represent the expected utilities of B and A in $\langle B, R, t\rangle$. Because λ and the state transition rules are measurable functions, it can be shown that u and z must exist and be integrable. Note that for $t \geqslant \bar{t}$, $u(t)$ and $z(t)$ are given by (5) and (6). We now proceed to characterize these functions on the interval $[0, \bar{t})$.

PROPOSITION 6.1: *On the interval* $[0, \bar{t})$, $u(t) \geqslant u(\bar{t})$, $z(t) > 0$, *and* $u(t) + z(t) \leqslant 1 - t$.

PROOF: In subgame $\langle A, R, t\rangle$, Player A can make the delayed offer $u(\bar{t})$ at time $\bar{t}$ (equivalent to using the SPE strategy σ in subgame $\langle A, R, \bar{t}\rangle$) and receive the expected utility of $u(\bar{t})$ (as shown in Section 4). Hence, $u(t) \geqslant u(\bar{t})$. Furthermore, in equilibrium, A must make an offer no later than time $\bar{t}$, so that B has a positive probability of receiving an offer at time $t \in [\bar{t}, 1)$, in which case B receives at least $u(t) > 0$. This shows that $z(t) > 0$. That $u(t) + z(t) \leqslant 1 - t$ follows from the fact that total expected utility available in subgame $\langle A, R, t\rangle$ is at most $1 - t$. *Q.E.D.*

Suppose that Player B uses strategy λ. Let $U(p, t)$ denote A's expected utility in subgame $\langle A, R, t\rangle$ when he makes an immediate offer of p and follows λ thereafter. Let s be the arrival time of offer p, and let $U(p, t|[t_o, t_1])$ represent A's expected utility conditional on $s \in [t_o, t_1]$. If $t \in [0, t_o)$, we have $U(p, t|[t_o, t_1]) = U(p, t_o|[t_o, t_1])$. This follows from the fact that the conditional distribution of arrival times is identical in the two cases, and the fact that the time that the offer p was made is not part of the payoff-relevant state of the continuation game.

PROPOSITION 6.2: *For $p < 1$ and $t \in [0, \bar{t})$, the function $U(p, t)$ is strictly decreasing in t. Because $p = 1$ cannot be an equilibrium offer, it follows that within subgames beginning on $[0, \bar{t})$, all offers are made immediately.*

PROOF: For any $s \in (t, 1]$, we have $U(p, t) = (s - t)U(p, t|[t, s]) + (1 - s)U(p, t|[s, 1])$. But $(1 - s)U(p, t|[s, 1]) = (1 - s)U(p, s|[s, 1]) = U(p, s)$. Also, $U(p, t|[t, s]) > 0$, because $p < 1$ and $z(t') > 0$ for all $t' > t$. Therefore, $U(p, t) > U(p, s)$. That offers are made immediately follows from the fact that Player A's expected utility from making a delayed offer p at time s in subgame $\langle A, R, t\rangle$ is the same as that from an immediate offer p in subgame $\langle A, R, s\rangle$. *Q.E.D.*

Let $p(t)$ denote the offer indicated by λ in $\langle A, R, t\rangle$. Because λ must be a best response within Ω, we have

$$u(t) = U(p(t), t) = \max_p U(p, t). \tag{9}$$

Because $p(t) < 1$ in equilibrium, it follows that for $t < s$, $u(t) \geqslant U(p(s), t) > U(p(s), s) = u(s)$, which implies the following proposition.

PROPOSITION 6.3: *The function $u(t)$ is strictly decreasing on $[0, \bar{t})$.*

In $\langle B, P, t\rangle$, B will accept p in equilibrium if $p > u(t)$. This, together with the fact that u is strictly decreasing, implies that any equilibrium offer in subgame $\langle A, R, t\rangle$ can be at most $u(t)$. Define

$$T(p) \equiv \sup\{t | u(t) \geqslant p\},$$

the inverse-like function whose graph is obtained by filling in the discontinuities

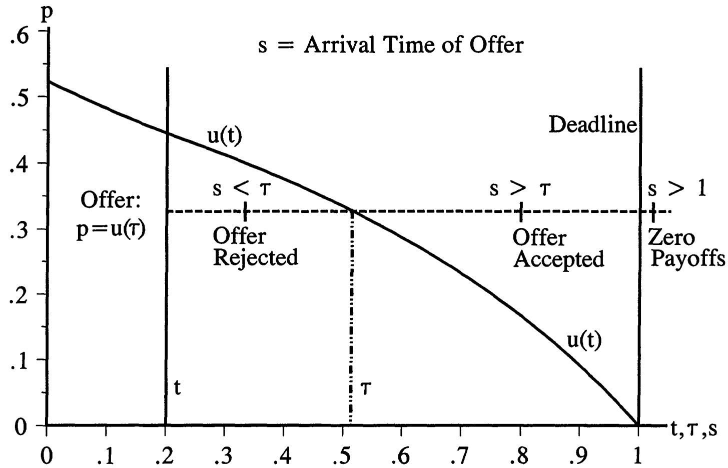

FIGURE 3.—An illustrative offer and response.

of $u(t)$ with vertical lines and reflecting it about the 45° line. Because u is strictly decreasing, it follows that $T(p)$ is continuous, $p > u(t)$ if $t > T(p)$, and $p < u(t)$ if $t < T(p)$. Since the offer will be accepted if it is received after $T(p)$, and rejected if received before $T(p)$, we call $T(p)$ the *acceptance horizon* of offer p (see Figure 3).

We are now in a position to prove the following proposition.

PROPOSITION 6.4: *The function $U(p,t)$ is given by*

$$U(p,t) = \int_t^{T(p)} z(s)\, ds + (1 - T(p))(1-p). \tag{10}$$

Furthermore, $U(p,t)$ and $u(t)$ are continuous, and $u = T^{-1}$.

PROOF: Suppose Player A makes an offer p in $\langle A, R, t\rangle$. Then if the offer arrives at time $s < T(p)$, we know that $p < u(s)$, so that in $\langle B, p, s\rangle$, B will reject p and A will get expected utility $z(s)$. Because the density function of the random delay is uniform, we have the first term on the right-hand side of (10). If the offer p arrives at $s > T(p)$, we know that $p > u(s)$, so that in $\langle B, p, s\rangle$, B will accept p and A will get $1-p$. Because the probability of this event is $1 - T(p)$, we have the second term on the right-hand side of (10).

The continuity of $U(p,t)$ follows from (10) and the continuity of $T(p)$. The continuity of $u(t)$ follows from (9), the continuity of $U(p,t)$, and the Theorem of the Maximum. Because u is continuous and strictly monotonic, T must be its inverse. *Q.E.D.*

Because there is a one-one correspondence in $\langle A, R, t\rangle$ between the relevant set of offers in $[0, u(t)]$, and the set of horizons for those offers in $[t, 1]$, we can formulate a player's optimization problem in terms of choice over the set of horizons. Let $\hat{U}(\tau, t)$ be the expected utility of Player A when he offers $u(\tau)$ in $\langle A, R, t\rangle$, so that $\hat{U}(\tau, t) \equiv U(u(\tau), t)$. From (10) and the fact that $T(u(\tau)) = \tau$, we have

$$\hat{U}(\tau, t) = \int_t^\tau z(s)\, ds + (1-\tau)(1-u(\tau)). \tag{11}$$

Let $\tau(t) \equiv T(p(t))$ be the acceptance horizon for the equilibrium offer $p(t)$, so that from (9) we have

$$u(t) = \hat{U}(\tau(t), t) = \max_{t \leqslant \tau \leqslant 1} \hat{U}(\tau, t). \tag{12}$$

By (11) we have that $u(t)$ satisfies

$$u(t) = \int_t^{\tau(t)} z(s)\, ds + (1-\tau(t))(1-u(\tau(t))), \tag{13}$$

and by an argument similar to the demonstration of (10), we have

$$z(t) = \int_t^{\tau(t)} u(s)\, ds + (1-\tau(t))u(\tau(t)). \tag{14}$$

We are now ready for the fundamental proposition of this section.

PROPOSITION 6.5: *For $t < \bar{t}$, $u(t) > (1-t)/(2-t)$ and $t < \tau(t) \leqslant \bar{t}$.*

PROOF: Part (c) of the proof of Proposition 4.6 implies that $\hat{U}(\bar{t}, \bar{t}) > \hat{U}(\tau, t)$ for $\tau > \bar{t}$. From (11) it is clear that this inequality implies $\hat{U}(\bar{t}, t) > \hat{U}(\tau, t)$ for $t \leqslant \bar{t} < \tau$, so that $\tau(t) \leqslant \bar{t}$. Because $\tau(t) = t$ implies that $u(t) = (1-t)/(2-t)$, it remains only to demonstrate that $u(t) > (1-t)/(2-t)$. Choose $t < \bar{t}$. To start, we show that $z(t) > z(\bar{t})$. Look at the integral on the right-hand side of (14). Because $u(s)$ is decreasing in s, it follows that $u(\tau(t))$ is a lower bound for the integrand, so that

$$\begin{aligned} z(t) &\geqslant [\tau(t) - t]u(\tau(t)) + (1-\tau(t))u(\tau(t)) \\ &= (1-t)u(\tau(t)) > (1-\bar{t})u(\bar{t}) = z(\bar{t}). \end{aligned}$$

Next, we see that (11), (12), and (13) give us

$$u(t) \geqslant \hat{U}(\bar{t}, t) = \int_t^{\bar{t}} z(s)\, ds + (1-\bar{t})(1-u(\bar{t})) = \int_t^{\bar{t}} z(s)\, ds + u(\bar{t}).$$

Since $z(s)$ in the integrand of the right-hand expression is strictly greater than the constant $z(\bar{t})$, we have

$$u(t) > z(\bar{t})(\bar{t} - t) + u(\bar{t}).$$

We proceed to show that $u(t) - (1-t)/(2-t) > 0$. From the above we have

$$u(t) - \frac{1-t}{2-t} > z(\bar{t})(\bar{t} - t) + u(\bar{t}) - \frac{1-t}{2-t}.$$

It is straightforward to confirm that the right-hand side of the inequality is zero when $t=\bar{t}$. Furthermore, that this expression has a negative derivative for $t<\bar{t}$ follows from the fact that

$$z(\bar{t})=\frac{(1-\bar{t})^2}{(2-\bar{t})}=\frac{1}{(2-\bar{t})^2}$$

where the first equality comes from (6) and the second from the definition of $\bar{t}$. Consequently, the right-hand side of the inequality must be positive for $t<\bar{t}$. *Q.E.D.*

PROPOSITION 6.6: *$\tau(t)=\bar{t}$ for all $t\in[0,\bar{t})$.*

PROOF: Choose $t<\bar{t}$. By Proposition 6.5, $\tau(t)\leqslant\bar{t}$. Suppose $\tau(t)\equiv t^*<\bar{t}$. Then, $\hat{U}(t^*,t)\geqslant\hat{U}(\tau,t)$ for all $\tau\in[t^*,\bar{t}]$. From (11) we have

$$\int_t^{t^*}z(s)\,ds+(1-t^*)(1-u(t^*))\geqslant\int_t^{\tau}z(s)\,ds+(1-\tau)(1-u(\tau)),$$

so that

$$(1-t^*)(1-u(t^*))\geqslant\int_{t^*}^{\tau}z(s)\,ds+(1-\tau)(1-u(\tau)).$$

Thus, $\hat{U}(t^*,t^*)\geqslant\hat{U}(\tau,t^*)$ for $\tau\in[t^*,\bar{t}]$. It follows that

$$u(t^*)=\hat{U}(t^*,t^*)=(1-t^*)/(2-t^*),$$

which contradicts the first inequality of Proposition 6.5. *Q.E.D.*

PROPOSITION 6.7: *On the interval $[0,\bar{t})$, the functions $u(t)$ and $z(t)$ are given by*

(15) $$u(t)=k_1e^t+k_2e^{-t}$$

and

(16) $$z(t)=k_2e^{-t}-k_1e^t$$

where

(17) $$k_1=\frac{\bar{t}(1-\bar{t})}{2(2-\bar{t})}e^{-\bar{t}}\doteq .04126$$

and

(18) $$k_2=\frac{1-\bar{t}}{2}e^{\bar{t}}i\doteq .48228.$$

PROOF: Applying $\tau(t)=\bar{t}$ to (13) and (14) and differentiating with respect to t yields $u'=-z$ and $z'=-u$ on $[0,\bar{t})$. Solving these differential equations yields (15) and (16). From (13), (14) and the fact that $\tau(t)=\bar{t}$ on $[0,\bar{t})$ it follows that $u(t)\to u(\bar{t})$ and $z(t)\to z(\bar{t})$ as $t\to\bar{t}$ from below. Equating $u(\bar{t})$ and $z(\bar{t})$ in

(15) and (16) to $u(\bar{t})$ and $z(\bar{t})$ in (5) and (6), and then solving for k_1 and k_2, yields the above values for these constants. *Q.E.D.*

The next proposition is now obvious:

PROPOSITION 6.8: *If λ is an SMPE strategy, then in $\langle A, p, t\rangle$, with $t \geqslant 0$ and with $u(t)$ as identified by Propositions 4.5 and 6.7, λ requires the following moves:*
(i) *if $p > u(t)$, accept offer p;*
(ii) *if $p < u(t)$ and $0 \leqslant t < \bar{t}$, reject p and offer $u(\bar{t})$ in $\langle A, R, t\rangle$;*
(iii) *if $p < u(t)$ and $\bar{t} \leqslant t < 1$, reject p and offer $u(t)$ in $\langle A, R, t\rangle$.*

Proposition 6.8 demonstrates that λ may differ from σ only in subgames $\langle Q, p, t\rangle$ for which $p = u(t)$. Again, the exceptional subgames as a class have a zero probability of being reached no matter what strategies the players follow.

7. A SYMMETRIC MARKOV-PERFECT EQUILIBRIUM ON THE INTERVAL [0, 1]: EXISTENCE

We now show that σ constitutes an SMPE. That is, we prove that for $0 \leqslant t < \bar{t}$, A's best response against σ is σ itself.

PROPOSITION 7.1: *Suppose B uses σ. Then A obtains expected utility $u(t)$ if he uses σ in subgame $\langle A, R, t\rangle$ with $0 \leqslant t < \bar{t}$.*

PROOF: First, we express $u(t)$, $t < \bar{t}$, in terms of $\bar{t}$ and $u(\bar{t})$. From the definitions of k_1, k_2, and $\bar{t}$, we know that

$$k_1 e^{t} - k_2 e^{-t} = u(t) \quad \text{and} \quad k_2 e^{t} - k_1 e^{-t} = (1 - \bar{t}) u(\bar{t}).$$

Using the above two equations to solve for k_1 and k_2 in terms of $\bar{t}$ and $u(\bar{t})$, we can substitute them into (15) and write it as

$$u(t) = \tfrac{1}{2} u(\bar{t}) \left[(2 - \bar{t}) e^{(\bar{t} - t)} + \bar{t} e^{-(\bar{t} - t)} \right]. \tag{19}$$

We now compute A's expected utility in $\langle A, R, t\rangle$ when both players use σ. Notice that there will never be an acceptance before $\bar{t}$. Moreover, if there is an acceptance (after $\bar{t}$), a player either gets $u(\bar{t})$ or $(1 - u(\bar{t}))$. More precisely, in $\langle A, R, t\rangle$, A obtains $u(\bar{t})$ if there is an acceptance after an even number of offers, or $1 - u(\bar{t})$ for an acceptance after an odd number of offers. When the random delay is uniformly distributed, the probability that in a sequence of n offers, the last offer arrives at time $1 \geqslant s_n \geqslant \bar{t}$ and all earlier offers arrive at $s_i < \bar{t}$ is

$$\int_t^{\bar{t}} \int_{s_1}^{\bar{t}} \cdots \int_{s_{n-2}}^{\bar{t}} \int_{\bar{t}}^{1} ds_n \, ds_{n-1} \cdots ds_1 = \frac{(1 - \bar{t})(\bar{t} - t)^{n-1}}{(n-1)!}.$$

Hence, A's expected utility in $\langle A, R, t\rangle$ from σ is

$$(1-\bar{t})\left(1+\frac{(\bar{t}-t)^2}{2!}+\frac{(\bar{t}-t)^4}{4!}+\cdots\right)(1-u(\bar{t}))$$
$$+(1-\bar{t})\left(\frac{(\bar{t}-t)^1}{1!}+\frac{(\bar{t}-t)^3}{3!}+\cdots\right)u(\bar{t}).$$

From the fact that the terms inside the large brackets in the above expression are respectively $\cosh(\bar{t}-t)$ and $\sinh(\bar{t}-t)$, we can simplify it to the right-hand side of (19). *Q.E.D.*

PROPOSITION 7.2: *For $t \geqslant 0$, σ is an SMPE strategy.*

PROOF: We need to show that for any $0 \leqslant t \leqslant \bar{t}$, σ is a best response against σ. So consider a subgame $\langle A, R, t\rangle$, let λ be a strategy for A on that subgame, and let $U(\lambda, t)$ be A's expected utility when he uses λ. By Proposition 7.1, we need show only that $U(\lambda, t) \leqslant u(t)$. Because we already know that σ is a best response against σ on subgames beginning in $[\bar{t}, 1]$, we can restrict our consideration to those λ that are identical to σ after time $\bar{t}$.

We begin with the case in which λ requires an offer $u(\hat{\tau}) > u(t)$. Suppose A makes an immediate offer $u(\tau)$ and uses λ thereafter, and let $U(\tau, \lambda, t)$ denote A's expected utility. We have

$$U(\lambda, t) \leqslant \max_{\tau \in [t, \bar{t}]} U(\tau, \lambda, t)$$

and it remains to show that the right-hand side of this inequality is no greater than $u(t)$. So assume that the offer $u(\tau)$ arrives at time s_1. If $s_1 \geqslant \tau$, B accepts the offer, and A obtains $(1-u(\tau))$. Otherwise, B rejects it and immediately makes a counteroffer $u(\bar{t})$. If A receives the counteroffer at some time $s_2 \geqslant \bar{t}$, he gets $u(\bar{t})$ because $\lambda = \sigma$ here; if $s_2 < \bar{t}$, A's expected utility is given by $U(\lambda, s_2)$. Hence, we can write

$$U(\tau, \lambda, t) = \int_t^{\tau}\left[\int_{s_1}^{\bar{t}} U(\lambda, s_2)\, ds_2 + \int_{\bar{t}}^{1} u(\bar{t})\, ds_2\right] ds_1 \tag{20}$$
$$+(1-\tau)(1-u(\tau)).$$

Differentiating (20) with respect to τ, we obtain

$$\frac{\partial U}{\partial \tau} = \int_{\tau}^{\bar{t}} U(\lambda, s_2)\, ds_2 + (1-\bar{t})u(\bar{t}) - (1-\tau)u'(\tau) - (1-u(\tau)) \tag{21}$$

and

$$\frac{\partial^2 U}{\partial \tau^2} = -U(\lambda, \tau) - (1-\tau)u''(\tau) + 2u'(\tau). \tag{22}$$

Since $u' < 0$ and $u'' = u > 0$ for $t < \bar{t}$, (22) is strictly negative. From the definition of u and $\bar{t}$, it is easily shown that $\partial U/\partial\tau = 0$ at $\tau = \bar{t}$. Thus, when played against σ, λ is dominated by any strategy that requires A to offer $u(\bar{t})$ in $\langle A, R, t\rangle$, and it follows that

$$(23) \qquad U(\lambda,t) \leqslant \int_t^{\bar{t}}\left[\int_{s_1}^{\bar{t}} U(\lambda,s_2)\,ds_2 + \int_{\bar{t}}^1 u(\bar{t})\,ds_2\right] ds_1 + (1-\bar{t})(1-u(\bar{t})).$$

Inequality (23) is true for all $t < \bar{t}$. Hence we can apply (23) for $t = s_2$. After replacing t in (23) by s_2, we obtain

$$(24) \qquad U(\lambda,s_2) \leqslant \int_{s_2}^{\bar{t}}\left[\int_{s_3}^{\bar{t}} U(\lambda,s_4)\,ds_4 + \int_{\bar{t}}^1 u(\bar{t})\,ds_4\right] ds_3 + (1-\bar{t})(1-u(\bar{t})).$$

Substituting the above for $U(\lambda, s_2)$ in (23), and after simplifying, we have

$$\begin{aligned}(25) \qquad U(\lambda,t) \leqslant{} & \int_t^{\bar{t}}\int_{s_1}^{\bar{t}}\int_{s_2}^{\bar{t}}\int_{s_3}^{\bar{t}} U(\lambda,s_4)\,ds_4\,ds_3\,ds_2\,ds_1 \\ & + (1-\bar{t})\left(1 + \frac{(\bar{t}-t)^2}{2!}\right)(1-u(\bar{t})) \\ & + (1-\bar{t})\left(\frac{(\bar{t}-t)^1}{1!} + \frac{(\bar{t}-t)^3}{3!}\right)u(\bar{t}).\end{aligned}$$

By successively applying the same substitution for $U(\lambda, s_4)$, $U(\lambda, s_6)$ and so on until $U(\lambda, s_n)$, we have

$$\begin{aligned}(26) \qquad U(\lambda,t) \leqslant{} & \int_t^{\bar{t}}\int_{s_1}^{\bar{t}} \cdots \int_{s_{2n-2}}^{\bar{t}}\int_{s_{2n-1}}^{\bar{t}} U(\lambda,s_{2n})\,ds_{2n} \cdots ds_1 \\ & + (1-\bar{t})\left(1 + \frac{(\bar{t}-t)^2}{2!} + \cdots + \frac{(\bar{t}-t)^n}{n!}\right)(1-u(\bar{t})) \\ & + (1-\bar{t})\left(\frac{(\bar{t}-t)^1}{1!} + \frac{(\bar{t}-t)^3}{3!} + \cdots + \frac{(\bar{t}-t)^{n+1}}{(n+1)!}\right)u(\bar{t}).\end{aligned}$$

But the sum of the last two terms in the above expression converge to $u(t)$ and the first term converges to zero. Hence, we have $U(\lambda, t) \leqslant u(t)$.

A similar argument for the case in which λ requires an offer $u(\hat{\tau}) \leqslant u(t)$ completes the proof. *Q.E.D.*

8. EQUILIBRIUM ON THE INTERVAL $[T_o, 1]$

We show that in subgames beginning at times $t < 0$, an SMPE strategy requires a player to accept any received offer greater than $u(0)$, and to reject any received offer less than $u(0)$ and make a delayed counteroffer of $u(\bar{t})$ at time 0.

PROPOSITION 8.1: *In any SMPE, Player A must obtain at least $u(0)$ in subgames $\langle A, p, t\rangle$ with $t<0$. Corollary: In any SMPE, Player A rejects the offer $p<u(0)$ in $\langle A, p, t\rangle$ if $t<0$.*

PROOF: In subgames $\langle A, p, t\rangle$, $t<0$, Player A can make the delayed offer $u(\bar{t})$ at time 0 and obtain expected utility $u(0)$. Consequently A will not accept less in any SMPE. *Q.E.D.*

PROPOSITION 8.2: *In an SMPE, Player A never makes an offer $p \geqslant u(0)$ in $\langle A, R, t\rangle$, $t<0$.*

PROOF: Suppose Player A makes an offer $p \geqslant u(0)$ at $t<0$. We compute an upper bound to his payoff. Since $p \geqslant u(0)$, and since $u(t)$ is monotonically decreasing on $[0,1]$, such an offer will be accepted if it arrives after time 0. In this case Player A obtains at most $1-u(0)$. Next, suppose the offer $p \geqslant u(0)$ arrives before time 0. Proposition 8.1 says that Player B must obtain at least $u(0)$ in all subgames before time 0. Hence Player A can get at most $1-u(0)$. We conclude that in any circumstances A gets at most $1-u(0)$ from the offer $p \geqslant u(0)$. Since $u(0)>1/2$, Proposition 8.1 implies that an offer of $p \geqslant u(0)$ will never be made in equilibrium. *Q.E.D.*

PROPOSITION 8.3: *In any SMPE Player A never makes an offer $p<u(0)$ at $t<0$.*

PROOF: First, suppose that Player A makes an offer $p<u(0)$ at $t \leqslant -1$. From the Corollary to Proposition 8.1, we know that Player B will reject such an offer, since with probability 1 this offer arrives before time 0. But Proposition 8.1 also says that in any SMPE, Player B gets at least $u(0)$ in all subgames $\langle B, R, t\rangle$, $t<0$. Therefore we conclude that Player A's maximum expected utility cannot be higher than $1-u(0)<u(0)$. This contradicts Proposition 8.1.

Next, suppose that Player A makes an offer $p<u(0)$ at time $-1<t<0$. Note that the offer will not be accepted unless it arrives at or after time $\tau=u^{-1}(p)$. If the offer arrives before time 0, then by Proposition 8.1 Player A gets at most $1-u(0)$. If the offer arrives at time s, $0 \leqslant s<\tau$, A gets $z(s)$. If the offer arrives after time τ, it will be accepted and A gets $1-p \equiv 1-u(\tau)$. The probability the offer will arrive before 0 is $-t$; the probability it will arrive after τ is $1+t-\tau$ (of course it must arrive no later than $1+t$). Thus an upper bound to his expected utility from this offer is given by

$$(-t)(1-u(0))+\int_0^\tau z(s)\,ds+(1+t-\tau)(1-u(\tau)).$$

However, if Player A makes the same offer at time 0, his expected utility is

$$\int_0^\tau z(s)\,ds + (1-\tau)(1-u(\tau)).$$

The difference between this expression and the previous one is:

$$\begin{aligned}&(1-\tau)(1-u(\tau)) + t(1-u(0)) - (1+t-\tau)(1-u(\tau))\\&\quad = t(1-u(0)) - t(1-u(\tau)) = t(u(\tau)-u(0))\end{aligned}$$

and the last expression is positive because both t and $u(\tau)-u(0)$ are negative. Thus the expected utility earned from an offer $p<u(0)$ made at time 0 is greater than that from the same offer made at $t<0$. *Q.E.D.*

From Propositions 8.2 and 8.3 we immediately have Proposition 8.4.

PROPOSITION 8.4: *If λ is an SMPE strategy, then in subgames $\langle A,p,t\rangle$, $T_o \leqslant t < 1$, λ requires the following moves*:

(a) *if $T_o \leqslant t < 0$ and $p > u(0)$, accept offer p*;

(b) *if $T_o \leqslant t < 0$ and $p < u(0)$, reject p, and in $\langle A,R,t\rangle$ make the delayed offer $u(0)$ at time $\hat{t}=0$*;

(c) *if $0 \leqslant t < 1$, play the strategy described in Proposition* 6.8.

Proposition 8.4 demonstrates that λ may differ from σ only in subgames $\langle Q,p,t\rangle$ for which $p=u(0)$. Such subgames are off the equilibrium path with respect to both size and timing of offers; furthermore, taken as a class, these subgames form a set of measure zero.

Finally, it is straightforward to adapt Propositions 8.2 and 8.3 to prove that σ is a best response against itself in subgames $\langle A,p,t\rangle$, $t<0$.

9. CONCLUSION

Recent strategic bargaining models focus on time preferences, bargaining horizons, information structures, sequencing of offers and outside opportunities as major research topics (see Roth (1985) and Rubinstein (1985b) for surveys). Asymmetric information is thought to be an important explanation for the phenomenon of delays in bargaining agreements (see Fudenberg and Tirole (1983), Fudenberg, Levine, and Tirole (1985), Hart (1989), and Rubinstein (1985a)).

In this paper, we have constructed a complete-information bargaining model whose equilibrium reflects stylized facts that describe bargaining under deadlines: initial delay, moderate offers, rejected offers, agreements near the deadline, and some failures in reaching agreement. Our interest was not to make the model highly realistic, but rather to explore the type of abstract framework that is consistent with generally observed bargaining behavior. Our model is one of complete information, but with imperfect player control over the timing of offers. In particular, we postulate that the required transmission time for offers is a random variable.

Let us again emphasize that the imperfect-control assumption is crucial to our results. This model characteristic falsifies the usual implications of backward induction in most complete-information bargaining models. When a random delay is associated with each offer, the size of an offer determines only the probability of its acceptance, not the sure reaction of the opponent. The strategic tradeoff in our model is between smaller offers with higher probabilities of rejection and larger offers with lower probabilities of rejection.

Notice, however, that the random-delay assumption serves our purpose only if players cannot circumvent these strategic tradeoffs. The following artificial modification of our model can illustrate this point. Suppose we allow a player to make the size of an offer contingent on the realization of its random delay. That is, suppose players are permitted to specify the size of offers in the form $\bar{p}(\bar{\delta})$, where $\bar{p}$ is any measurable function and $\bar{\delta}$ is the realized random delay. One can show that in a Markov-perfect equilibrium of this game, the first player to move makes the delayed offer $\bar{p}(\bar{\delta}) \equiv 1 - e^{\bar{\delta}-1}$ at time $t = 0$. Moreover, the equilibrium utility for Player Q in subgames $\langle Q, R, t \rangle$ is given by $\bar{p}(t)$, $0 \leqslant t \leqslant 1$. In equilibrium the offer $\bar{p}(s)$ will be accepted when it arrives at time s. This outcome is qualitatively similar to those found in the literature: in equilibrium, the first offer ever made will be accepted. Any delay in agreement is due to the strategic delay assumption and the (random) transmission time of the first offer. Essentially, if a player's strategy can effectively eliminate the uncertainty associated with random delays so that he can precisely predict the opponent's (expected) utility from rejecting the offer, then once an equilibrium offer is made, it will be accepted.

It would be interesting to study the above model with a general probability distribution of random delays. It turns out that the problem is rather intractable if time and random delays are assumed continuous. Thus it remains an open question whether or not the qualitative properties of the equilibrium derived in this paper (initial stalling, offers that may be rejected, agreements reached late in the game, fairly even equilibrium divisions of surplus, and an occasional failure to reach an agreement) remain true for arbitrary probability distributions of random delays.[18]

Another important topic for future research is the voluntary adoption of deadlines. Given that deadlines are very common, it will be interesting to investigate how they are actually chosen. Upon the adoption of a deadline, what are the incentives to commit to it? Our model here can be regarded as a first step in the study of bargaining procedures as a strategic decision.

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[18] We have developed a computer algorithm, available from the authors, to compute subgame-perfect equilibria for discrete versions of the game with arbitrary distributions of random delays. The simulations we have tried conform to the pattern described here.

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