Abstraction and Control for Groups of Robots

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***Abstract*—This paper addresses the general problem of controlling a large number of robots required to move as a group. We propose an abstraction based on the definition of a map from the configuration space Q of the robots to a lower dimensional manifold A, whose dimension is independent of the number of robots. In this paper, we focus on planar fully actuated robots. We require that the manifold A has a product structure $A = G \times S$, where G is a Lie group, which captures the position and orientation of the ensemble in the chosen world coordinate frame, and S is a shape manifold, which is an intrinsic characterization of the team describing the "shape" as the area spanned by the robots. We design decoupled controllers for the group and shape variables. We derive controllers for individual robots that guarantee the desired behavior on A. These controllers can be realized by feedback that depends only on the current state of the robot and the state of the manifold A. This has the practical advantage of reducing the communication and sensing that is required and limiting the complexity of individual robot controllers, even for large numbers of robots.**

***Index Terms*—Abstraction, control, Lie group, shape.**

I. Introduction

THERE has been a great deal of interest in cooperative robotics in the last few years, triggered mainly by the technological advances in control techniques for single vehicles and the explosion in computation and communication capabilities. The research in the field of control and coordination for multiple robots is currently progressing in areas like automated highway systems, formation flight control, unmanned underwater vehicles, satellite clustering, exploration, surveillance, search and rescue, mapping of unknown or partially known environments, distributed manipulation, and transportation of large objects.

In this paper, we consider the problem of controlling a large number of robots required to move as a team from an initial to a final region of the space. For example, consider the problem of moving 100 planar robots with arbitrary initial positions through a tunnel while staying grouped so that the distance between each pair does not exceed a certain value. The simplest approach involves reference trajectories and control laws for each robot to stay on the designed trajectory. While this is obviously feasible [2], [22], it is intractable from a computational viewpoint. As the number of robots increase, it is desirable to have a certain level of *abstraction*. The motion generation/control problem should be solved in a lower dimensional space which captures the behavior of the group and the nature of the task.

One possible way of accomplishing this is to require the robots to conform to one or more rigid *virtual structures*. In this case, the motion-planning problem is reduced to a left invariant control system on $SE(3)$ (or $SE(2)$ in the planar case), and the individual trajectories are $SE(3)$ [$SE(2)$] orbits [2], [3]. The literature on stabilization and control of virtual structures is rather extensive. Most of the recent works model formations using *formation graphs*, which are graphs whose nodes capture the individual agent kinematics or dynamics, and whose edges represent interagent constraints that must be satisfied [4], [13], [17], [20]. Characterizations of rigid formations can be found in [2] and [7]. The controllers guaranteeing local asymptotic stability of a given rigid formation can be derived using standard techniques such as input–output linearization [4], input-to-state stability [21], or Lyapunov energy-type functions. Examples of such functions include positive definite convex *formation functions* [5], [13] and biologically inspired *artificial potential functions* [12]. The global minima of such functions exhibit $SE(l)$, $l = 1, 2, 3$ symmetry and expansion/contraction symmetries, which can be used to decouple the mission-control problem into a formation-keeping subproblem and a maneuver subproblem [12].

The same idea of artificial potential functions is used by scientists studying behavior-based control [1] and swarming-type behaviors [6]. In [14], the authors consider a distributed control approach for groups of robots, called the social potential fields method, which is based on artificial spring force laws between individual robots and robot groups. Interesting simulation results are included, but it is difficult to obtain proofs of convergence with such approaches. A continuous-time model for swarm aggregation is presented in [8], where it is proved that a group of agents form a cohesive swarm if each pair is subject to a potential function which is attractive for large distances and repulsive for small distances. The geometric pattern formation in swarms is approached in discrete time as well. In [18], the authors propose a simple distributed heuristic algorithm for convergence to a circle, while in [19], algorithms for converging to a single point are presented.

It is worth reviewing some of the limitations of the approaches adopted in previous work. Imposing the constraint of a virtual structure is practical in small groups, since the complexity of the graph underlying it grows exponentially [4]. The optimal design of trajectories becomes prohibitively complicated, even for small teams of robots [2]. Moreover, virtual structures might unnecessarily constrain the problem. Second, leader–follower architectures, such as the one proposed in [4], require identification and ordering of the robots. This makes the team behavior sensitive to failures. For example, if the

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team leader has a failure, any exception-handling scheme must impact all of the robots that follow the leader. Finally, imposing rigidity constraints [7] induces an inherent coupling between the control systems on the symmetry group and the shape space. For example, in [12], the authors have to limit the speed of convergence on the symmetry group so that, while moving as a group, the individual agents do not leave the local regions of attractions guaranteeing convergence to the desired shape.

In this paper, we attempt to derive a formal abstraction for a team of robots that can be used to control the position, orientation, and shape of the team. The abstraction is based on the definition of a map ϕ from the configuration space Q of the robots to a lower dimensional manifold A, independent of the number and ordering of the robots. We require that the manifold A has a product structure $A = G \times S$, where G is a *Lie group* which captures the dependence of the ensemble on the chosen world coordinate frame and S is a *shape manifold*, which is an intrinsic description of the team. In addition, we impose that the shape variables $s \in S$ and the group variables $g \in G$ be controlled independently, so that the user can easily command the independent variables. For example, the user can change the shape of the formation without modifying the group trajectory and vice versa.

In order to ensure that the control computations scale well with the number of robots, we require that each robot only have access to its own state and the abstract state $a \in A$. From a practical standpoint, it is easy for each robot to carry sensors that yield estimates of its own state. The difficulty arises in estimating the group and shape of the entire formation. Measurements from overhead sensors (e.g., cameras on unmanned aerial vehicles (UAVs) traveling with the robots) that are broadcast over a wireless network can be used to realize this feedback. While this communication architecture is easily classified as centralized and the UAV needs to determine the state of all robots, each ground robot does not need explicit information about all the others, and the amount of data sent to the robots by the UAV is small and does not scale with the number of robots.

In this paper, we restrict our attention to fully actuated mobile robots in the plane so that G is a subset of $SE(2)$. $g \in G$ provides the position and orientation of the group reference frame. The shape variables $s \in S$ describe the distribution of the robots in the group reference frame. The problem is formulated in Section II, and our geometric approach to solving it is outlined in Section III. Sections IV–VI define an abstraction, discuss its significance, and show that the requirements of the problem formulated in Section II are satisfied. Illustrative simulation results are included in Section VII. The paper concludes with a statement of contributions and future work in Section VIII.

II. Definitions and Problem Formulation

Consider N kinematically controlled robots with states q_i belonging to manifold Q_i and control spaces U_i. For planar fully actuated agents, the states are position vectors $q_i \in Q_i = \mathbb{R}^2$, $i = 1, \ldots, N$ with respect to some world frame $\{W\}$, and the controls $u_i \in U_i = \mathbb{R}^2$ as follows:

$$\dot{q}_i = u_i. \tag{1}$$

Collecting all of the robot states together, we obtain a $2N$-dimensional control system

$$\dot{q} = u \tag{2}$$

where $q \in Q = \prod_{i=1}^{N} Q_i = \mathbb{R}^{2N}$, $u \in U = \prod_{i=1}^{N} U_i = \mathbb{R}^{2N}$.

The motion (behavior) of the ensemble of robots is determined if the corresponding velocities are specified as follows.

Definition 1: (Behavior) Any vector field $X_Q \in TQ$ is called a behavior.

Given a large number of robots evolving on the configuration space Q, we want to be able to solve motion-generation/control problems on a smaller dimensional space, which captures the essential features of the group, according to the class of tasks to be accomplished. We want the dimension of the control problem to be independent of the number of agents and independent of the possible ordering of the robots. These requirements will provide good scaling properties and control laws which are robust to individual failures.

We also need to make sure that, after solving the task on the small dimensional space, we can go back and generate control laws for the individual agents. All of these ideas lead to the following definitions.

Definition 2: (Abstraction) Any surjective submersion

$$\phi : Q \to A, \quad \phi(q) = a \tag{3}$$

is called an abstraction if it is invariant to permutations of the robots and the dimension n of A is not dependent on the number of robots N. A and a are called abstract manifold and abstract state, respectively.

It is assumed that the abstract state a is physically significant in accordance with the task to be accomplished.

In addition, if possible, it is desired that A have a product structure

$$A = G \times S, \quad a = (g, s), \quad \phi = (\phi_g, \phi_s) \tag{4}$$

where G is a Lie group. An arbitrary $g \in G$ defines the gross position and orientation of the team in the world frame $\{W\}$, and it is called the *group* variable. $s \in S$ is called the *shape* variable. The main idea is to have a control-suited description of the team of robots a in terms of the pose g of a virtual structure, which captures the dependence of the team on the world frame $\{W\}$, plus a shape s, which is decoupled from g and, therefore, an intrinsic property of the formation. In other words, if $\bar{g}$ is an arbitrary element of G, we require the map ϕ to satisfy

$$\phi(q) = (g, s) \Rightarrow \phi(\bar{g}q) = (\bar{g}g, s) \tag{5}$$

where $\bar{g}q$ represents the block diagonal action of the group element $\bar{g}$ on the configuration $q \in Q$, and $\bar{g}g$ represents the left translation of g by $\bar{g}$ using the composition rule on the group G. Since we only approach planar robots in this paper, G is a subset of $SE(2)$. $\bar{g}q$ represents a rigid displacement of all of the robots by $\bar{g}$. Equation (5) shows that the map ϕ is left invariant, which gives invariance of our to-be-designed control laws to the pose of the world frame $\{W\}$. Indeed, if the world frame $\{W\}$ is displaced by $\bar{g}$, the shape s is not affected, while the pose g is left translated by $\bar{g}$.

Instead of designing high-dimensional behaviors X_Q, we want to be able to describe collective behaviors in terms of time-parameterized curves on the lower dimensional abstract manifold A.

Definition 3: (Abstract Behavior) Any vector field $X_A \in TA$ is called an abstract behavior.

Let $d\phi$ denote the differential (tangent) of the map ϕ. Note that the submersion condition in *Definition 2* guarantees the surjectivity of the differential $d\phi$ at any $q \in Q$, which will guarantee the existence of vector fields X_Q pushed forward to any abstract behavior X_A.

The abstraction ϕ gives a decomposition of the space of behaviors on Q into behaviors which can be "seen" in the abstract manifold A and behaviors which cannot be seen in A.

Definition 4: (Detectable behaviors) A behavior $X_Q \in TQ$ which is mapped to a nonzero abstract behavior $X_A \in TA$ is called a detectable behavior. A behavior which is not detectable is called nondetectable.

Our goal in this paper is to generate individual control laws which are mapped to desired abstract (collective) behaviors, i.e., wisely chosen low-dimension descriptions. Therefore, we will not allow individual motions which cannot be captured in A, because this would be a waste of energy. However, nondetectable behaviors can be useful to accommodate specifications which are not captured by A.

We are now able to formulate the main problem.

Problem 1: (Control) Determine physically meaningful formation abstractions ϕ, abstract behaviors X_A, and corresponding individual robot control laws u_i satisfying the following requirements:

1) the abstract state a is stationary if and only if all the robots q_i are stationary;
2) the abstract manifold A has a product structure (4) and ϕ satisfies the left invariance property (5);
3) the control systems on group G and shape S are decoupled;
4) if the state a of the abstract manifold is bounded, then the state of each robot q_i is bounded.

Requirement 1) from *Problem 1* guarantees that each individual motion on Q_i can be "seen" in the small dimensional manifold A and, therefore, can be "penalized" by control. This is equivalent to the detectability of the corresponding behavior X_Q. If requirements 2) and 3) are satisfied, then one can design control laws for the interest variables on a separately, e.g., change the pose of the formation g while preserving the shape s. Requirement 4) is self-explanatory.

In addition to the requirements explicitly formulated in *Problem 1*, it is desired that the energy spent by the individual robots to produce a desired abstract behavior be kept to a minimum. Also, the amount of interrobot communication in the overall control architecture should be limited.

Remark 1: The description above can be easily extended to accommodate underactuated robots with states q_i belonging to manifolds Q_i equipped with drift-free control distributions Δ_i: $Q_i \times U_i \rightarrow TQ_i$, where U_i is the control space and TQ_i is the tangent bundle of Q_i. Then, a similar large control system incorporating all of the individual underactuation constraints can be obtained by collecting all robot states $Q = \prod_{i=1}^{N} Q_i$ and defining a control distribution Δ obtained from the individual control distributions through direct sum $\Delta = \bigoplus_{i=1}^{N} \Delta_i$. The canonical projections are defined by π_i: $Q \rightarrow Q_i$, $\pi_i(q) = q_i$ and $d\pi_i$: $TQ \rightarrow TQ_i$, $d\pi_i(\Delta) = \Delta_i$. In this case, an abstract behavior should incorporate the underactuation constraints. They naturally arise on A by pushing forward the allowed control directions in Δ (or its accesibility algebra) through ϕ. The underactuated case will be studied in a future paper.

III. Approach

In this section, we characterize the solution to *Problem 1*. First, note that the map ϕ gives a foliation [9] of the configuration space Q. We assume that the abstract manifold has the desired product structure $A = G \times S$. Let Ω_g be the codistribution spanned by the differential forms obtained by differentiating each component of ϕ_g. Similarly, Ω_s is the codistribution determined by ϕ_s. Let Δ_g and Δ_s denote the corresponding annihilating distributions, i.e.,

$$\Omega_g(\Delta_g) = 0, \quad \Omega_s(\Delta_s) = 0. \tag{6}$$

Let $\bar{\Delta}_g$ be any distribution so that $\bar{\Delta}_g + \Delta_g = TQ$ and $\dim \bar{\Delta}_g + \dim \Delta_g = \dim Q$. Similarly, denote by $\bar{\Delta}_s$ any distribution so that $\bar{\Delta}_s + \Delta_s = TQ$ and $\dim \bar{\Delta}_s + \dim \Delta_s = \dim Q$. Then

$$X_Q \in \bar{\Delta}_g \tag{7}$$

guarantees that, on the abstract manifold, at $a = (g, s)$, g changes in time whenever q does. Similarly

$$X_Q \in \bar{\Delta}_s \tag{8}$$

corresponds to a change in the shape variable s. The set of detectable behaviors is given by $\bar{\Delta}_g + \bar{\Delta}_s$. Requirement 1) from *Problem 1* can therefore be written as

$$X_Q \in \bar{\Delta}_g + \bar{\Delta}_s. \tag{9}$$

In other words, system (2) is forbidden to move on a leaf $\phi = \mathrm{const.}$ (motion which could not be "observed" on the abstract manifold A) if and only if (9) is satisfied.

To formulate the decoupling condition between the control of group G and the shape S of A (item 3) of *Problem 1*), we first require that the distributions $\bar{\Delta}_g$ and $\bar{\Delta}_s$ be independent, i.e., $\bar{\Delta}_g \cap \bar{\Delta}_s = 0$, where 0 denotes the zero vector field. Then the decoupling condition is satisfied if the codistribution corresponding to g annihilates the visible motion corresponding to s and the other way around. Explicitly

$$\Omega_g(\bar{\Delta}_s) = 0, \quad \Omega_s(\bar{\Delta}_g) = 0. \tag{10}$$

This is easy to see if we differentiate $g = \phi_g(q)$ in coordinates to obtain $\dot{g} = d\phi_g \dot{q}$. If $\dot{q}$ is detectable [satisfies (9)], then we can write $\dot{q} = A_g u_g + A_s u_s$, where A_g and A_s are some matrices whose columns span $\bar{\Delta}_g$ and $\bar{\Delta}_s$. u_s does not affect $\dot{q}$ if and only if $d\phi_g A_s = 0$, which, if we go back to the coordinate-free representation, means $\Omega_g(\bar{\Delta}_s)$. Similar reasoning can be made for the shape s. u_g and u_s separately control g and s. They will be the actual controls for group and shape, after some convenient rescaling.

For 4), note that *Problem 1* can actually be seen as an input–output linearization problem [9] for the control system (2) with output $a = \phi(q)$. The total relative degree is $\dim(Q) - n$, since each robot is kinematically controlled. The vector field X_A guarantees some desired behavior of the output (which we call the abstract state) a, which will, of course, guarantee its boundness. Now the hardest problem, as usual in input–output linearization, is calculating and stabilizing the internal dynamics. This would imply, in general, finding the appropriate coordinate transformation separating the internal dynamics from output dynamics, calculating the corresponding zero dynamics, and studying its stability. To avoid this, we try to define the output map so that bounds on the output would easily imply bounds on the state, so it will not be necessary to explicitly calculate the internal dynamics.

Given a vector field $X_A \in TA$, the set of all vector fields $X_Q \in TQ$ which maps to X_A is underdetermined. For simplicity, let $\dot{q}$ and $\dot{a}$ denote the coordinates of X_Q and X_A, respectively. Then

$$d\phi \dot{q} = \dot{a}. \tag{11}$$

The usual way of solving the undetermined linear equation (11) is to find the minimum norm vector $\dot{q}$ satisfying it. Even though more general metrics (i.e., the kinetic energy metric [3]) can be considered, we assume that Q is equipped with a Euclidean metric. Then the solution to the minimization problem $\min_{\dot{q}} \dot{q}^T \dot{q}$ under constraint (11) is given by

$$\dot{q} = d\phi^T (d\phi d\phi^T)^{-1} \dot{a}. \tag{12}$$

Note that, since ϕ is a submersion, $\phi_1, \ldots, \phi_n$ are functionally independent or, equivalently, $d\phi$ is full-row rank, which implies that $d\phi d\phi^T$ is invertible. By writing $d\phi^T = (d\phi_g^T, d\phi_s^T)$, $a^T = (g^T, s^T)$, (12) becomes

$$\dot{q} = d\phi_g^T \left(d\phi_q d\phi_g^T\right)^{-1} \dot{g} + d\phi_s^T \left(d\phi_s d\phi_s^T\right)^{-1} \dot{s} \tag{13}$$

if $d\phi_g d\phi_s^T = 0$.

Note that $\dot{q}$ from (13) satisfies the detectability and decoupling conditions formulated in terms of distributions (9) and (10) if, in coordinates, $\bar{\Delta}_g$ and $\bar{\Delta}_s$ are spanned by $d\phi_g^T$ and $d\phi_s^T$, respectively. Indeed, the linear independence of $d\phi_g$ and $d\phi_s$ implies the independence of $\bar{\Delta}_g$ and $\bar{\Delta}_s$, and (10) is implied by $d\phi_g d\phi_s^T = 0$. Moreover, (6) implies that $\bar{\Delta}_g$ and Δ_g are orthogonal. The same is true for $\bar{\Delta}_s$ and Δ_s.

Finally, to limit the amount of interrobot communication in the overall control architecture, we want to achieve an architecture where the control law of a robot only depends on its own state and the low-dimensional state of the team from the group manifold, as follows:

$$u_i = u_i(q_i, a). \tag{14}$$

IV. Abstraction

In this section, we define a physically significant abstraction (3) and show that it satisfies requirements 1)–3) from *Problem 1*. Satisfaction of requirement 4) will be proved in Section V.

For an arbitrary configuration $q \in Q$, the group part g of the abstract state a is defined by $g = (R, \mu) \in G = SE(2)$. Let

$$\mu = \frac{1}{N} \sum_{i=1}^{N} q_i \in \mathbb{R}^2. \tag{15}$$

Define

$$r_i = [x_i, y_i]^T = R^T (q_i - \mu), \quad i = 1, \ldots, N. \tag{16}$$

The equation that we will use to define the rotational part $R \in SO(2)$ is

$$\sum_{i=1}^{N} x_i y_i = 0. \tag{17}$$

The precise definition of the rotation is given in coordinates in (29). In this paper, we restrict our attention to a two-dimensional (2-D) shape $s = [s_1, s_2]$ defined by

$$s_1 = \frac{1}{N-1} \sum_{i=1}^{N} x_i^2$$
$$s_2 = \frac{1}{N-1} \sum_{i=1}^{N} y_i^2. \tag{18}$$

Since $SO(2)$ is one-dimensional (1-D), the dimension of the abstract manifold A is $n = 5$, independent of the number of robots N. Also, it is obvious that our definitions (15), (17), and (18) of group and shape are invariant to permutations of robots, as required by *Definition 2*. The submersion condition will be studied later in this section.

Before we show that the abstraction ϕ defined above solves *Problem 1*, we study its physical significance.

A. Significance

There are two slightly different interpretations of the abstraction defined by (15)–(18). Let

$$\Sigma = \frac{1}{N-1} \sum_{i=1}^{N} (q_i - \mu)(q_i - \mu)^T \tag{19}$$
$$\Gamma = -(N-1) E_3 \Sigma E_3 \tag{20}$$

where E_3 is defined by (23). Note that, since $E_3^T = E_3^{-1} = -E_3$, Γ and $(N-1)\Sigma$ have the same eigenstructure.

1) Spanning Rectangle: μ and Γ in (15) and (20) can be seen as the centroid and inertia tensor of the system of particles q_i with respect to the centroid and orientation $\{W\}$. Let $\{M\}$ define a virtual frame with pose $g = (R, \mu)$ in $\{W\}$. Then r_i is the expression of $q_i - \mu$ in the virtual frame $\{M\}$. The rotation (17) defines the orientation of the virtual frame so that the inertia tensor of the system of points r_i in $\{M\}$ is diagonal. $(N-1)s_1$ and $(N-1)s_2$ are the eigenvalues of the tensor and are, therefore, measures of the spatial distribution of the robots along the axis of the virtual frame $\{M\}$.

It is interesting to note that the shape variables provide a bound for the region occupied by the robots. From (18), it immediately follows, for any $i = 1, \ldots, N$, that

$$|x_i| \leq \sqrt{(N-1)s_1}, \quad |y_i| \leq \sqrt{(N-1)s_2}. \tag{21}$$

The conclusion can be stated as follows. *An ensemble of N robots described by a five-dimensional (5-D) abstract variable $a = (g, s) = (R, \mu, s_1, s_2)$ is enclosed in a rectangle centered at μ and rotated by $R \in SO(2)$ in the world frame $\{W\}$. The sides of the rectangle are given by $2\sqrt{(N-1)s_1}$ and $2\sqrt{(N-1)s_2}$.*

We call the rectangle described by (R, μ, s_1, s_2) the *spanning rectangle*.

2) Concentration Ellipsoid: μ and Σ given by (15) and (19) can be interpreted as sample mean and covariance of a random variable with realizations q_i. If the random variable is known to be normally distributed, then, for a sufficiently large N, μ and Σ converge to the real parameters of the normal distribution. R in (17) is the rotation that diagonalizes the covariance, and s_1 and s_2 are the eigenvalues of the covariance matrix. This means that, for a large number of normally distributed robots, μ, R, s_1, and s_2 give the pose and semiaxes of a concentration ellipsoid.

Specifically, it is known that contours of constant probability p for normally distributed points in plane with mean μ and covariance Σ are ellipses described by [16]

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = c, \quad c = -2\ln(1-p). \tag{22}$$

The ellipse in (22), called the *equipotential* or *concentration ellipse*, has the property that p percent of the points are inside it and can be therefore used as a spanning region for our robots, under the assumption that they are normally distributed. Therefore, we can make the following statement: p percent of a large number N of normally distributed robots described by a 5-D abstract variable $a = (g, s) = (R, \mu, s_1, s_2)$ is enclosed in an ellipse centered at μ, rotated by $R \in SO(2)$ in the world frame $\{W\}$ and with semiaxes $\sqrt{cs_1}$ and $\sqrt{cs_2}$, where c is given by (22).

Even though the normal distribution assumption might seem very restrictive, we show in Section IV-C that it is enough that the robots be normally distributed in the initial configuration. Our controls laws will preserve the normal distribution.

3) Spanning Rectangle Versus Concentration Ellipsoid: The abstraction based on the spanning rectangle as defined in Section IV-A1 has the advantage that it provides a rigorous bound for the region occupied by the robots and does not rely on any assumption on the distribution of the robots. The main disadvantage is that this estimate becomes too conservative when the number of robots is large. Indeed, the lengths of the sides of the rectangle scale with $\sqrt{N-1}$, so for a large N the spanning rectangle might become very large, even though the robots might be grouped around the centroid μ.

On the other hand, the size of a concentration ellipsoid as defined in Section IV-A2 does not scale with the number of robots, which makes this approach very attractive for very large N. However, it has the disadvantage of assuming a normally distributed initial configuration of the team and does not provide a rigorous bound for the region occupied by the robots. Roughly speaking, $(1-p)N$ is left out of the p ellipse. Increasing p will decrease the number of the robots which be outside but will also increase the size of the ellipsoid.

To have an idea of what is a "large" number N for which the second approach is more feasible, note that the spanning rectangle and the rectangle in which the concentration ellipsoid is inscribed are similar and the ratio is $\sqrt{(N-1)/c}$. The ratio of their areas is therefore $(N-1)/c$. For example, if $p = 0.99$, we have $c = 9.2103$, and the spanning rectangle becomes larger for $N \geq 11$. If $N = 100$, the area of the spanning rectangle is 10.7488 larger than the area of the rectangle circumscribing the ellipse, and only one robot might be left out of the ellipse.

Remark 2: More shape variables can be defined by considering higher moments. For example, the fourth moment (kurtosis) might be used to quantify how uniformly the robots occupy the spanning region. More generally, a complete shape space invariant to translations and rotations can be defined using Jacobi coordinates. However, only a subset of this might be interesting from a formation control point of view. Examples include dot products and triple products, which translate to distances, angles, areas, and volumes [10], [15].

On Requirement 2) of *Problem 1*, we have the following.

Proposition 1: The abstraction ϕ defined by (15), (17), and (18) satisfies the left invariance property (5).

Proof: The proof is based on the invariance of the spectrum of a matrix to orthogonal transformations and is omitted. □

B. Detectable Behaviors and Decoupling of Group and Shape

In this section, under the assumption that the configuration space Q is equipped with a Euclidean metric, we construct detectable behaviors and decoupled control systems for group and shape, in accordance with requirements 1) and 4) from *Problem 1*.

To this end, we need to bring our definitions of formation variables (15), (17), and (18) to more convenient forms. Let

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$E_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{23}$$
$$H_1 = I_2 + R^2 E_2$$
$$H_2 = I_2 - R^2 E_2$$
$$H_3 = R^2 E_1 \tag{24}$$

where I_2 is the 2×2 identity matrix. Using a parameterization $R = [\cos(\theta) - \sin(\theta); \sin(\theta)\cos(\theta)]$, it is easy to see that the matrices H_i are symmetric and

$$\begin{aligned} H_1^2 &= 2H_1 \\ H_2^2 &= 2H_2 \\ H_3^2 &= I_2 \end{aligned} \tag{25}$$

$$\begin{aligned} H_1H_2 &= 0 \\ H_1H_3 &= H_3 - E_3 \\ H_2H_3 &= H_3 + E_3. \end{aligned} \tag{26}$$

Then, some simple calculations and the observation that E_3 is skew-symmetric show that (17) defining the rotational part becomes

$$\sum_{i=1}^{N}(q_i - \mu)^T H_3(q_i - \mu) = 0 \tag{27}$$

while the description of the shape (18) takes the form

$$\begin{aligned} s_1 &= \frac{1}{2(N-1)}\sum_{i=1}^{N}(q_i - \mu)^T H_1(q_i - \mu) \\ s_2 &= \frac{1}{2(N-1)}\sum_{i=1}^{N}(q_i - \mu)^T H_2(q_i - \mu). \end{aligned} \tag{28}$$

Let the rotation $R \in SO(2)$ be parameterized by θ. If the amount of rotation is restricted to $\theta \in (-\pi/2, \pi/2)$, a unique solution of (27) is given by

$$\theta = \frac{1}{2}\text{atan2}\left(\sum_{i=1}^{N}(q_i-\mu)^T E_1(q_i-\mu), \sum_{i=1}^{N}(q_i-\mu)^T E_2(q_i-\mu)\right) \tag{29}$$

where we use the Fortran atan2 notation for the inverse tangent function, which, by definition, is restricted to take values in $(-\pi, \pi)$.

We now characterize the set of detectable behaviors (9) for the map ϕ given by (15), (29), and (28) together with definitions (23) and (24).

First note that

$$\begin{aligned} dH_1 &= 2H_3 d\theta \\ dH_2 &= -2H_3 d\theta. \end{aligned} \tag{30}$$

Using (15), (29), and (28), it follows that

$$ds_1 = \frac{1}{N-1}\sum_{i=1}^{N}(q_i - \mu)^T H_1 dq_i \tag{31}$$

$$ds_2 = \frac{1}{N-1}\sum_{i=1}^{N}(q_i - \mu)^T H_2 dq_i. \tag{32}$$

By differentiating (27) and using (15) and (28), we have

$$d\theta = \frac{1}{(N-1)(s_1 - s_2)}\sum_{i=1}^{N}(q_i - \mu)^T H_3 dq_i. \tag{33}$$

Then, the codistributions Ω_g and Ω_s, as defined in Section III, are given by

$$\Omega_g = \text{span}\{d\mu, d\theta\}, \quad \Omega_\rho = \text{span}\{ds_1, ds_2\} \tag{34}$$

and the control distributions corresponding to group $\bar{\Delta}_g$ and shape $\bar{\Delta}_s$ are given by

$$\bar{\Delta}_g = \text{span}\left\{X_q^\mu, X_q^\theta\right\} \tag{35}$$

$$\bar{\Delta}_s = \text{span}\left\{X_q^{s_1}, X_q^{s_2}\right\} \tag{36}$$

where

$$X_q^\mu = \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}, \quad X_q^\theta = \begin{bmatrix} H_3(q_1 - \mu) \\ \vdots \\ H_3(q_N - \mu) \end{bmatrix} \tag{37}$$

$$X_q^{s_1} = \begin{bmatrix} H_1(q_1 - \mu) \\ \vdots \\ H_1(q_N - \mu) \end{bmatrix}, \quad X_q^{s_2} = \begin{bmatrix} H_2(q_1 - \mu) \\ \vdots \\ H_2(q_N - \mu) \end{bmatrix}. \tag{38}$$

I_2 is the identity matrix.

Therefore, in accordance with (9), Requirement 1) of *Problem 1* is satisfied if we restrict the behaviors to the set $\bar{\Delta}_g + \bar{\Delta}_s$ given by (35)–(38).

We now show that the control distributions $\bar{\Delta}_g$ and $\bar{\Delta}_s$ are orthogonal, so decoupled control systems can be designed for group and shape, in accordance with Requirement 3) of *Problem 1*. Indeed, the two columns of X_q^μ are obviously orthogonal. It is easy to see that X_q^θ, $X_q^{s_1}$ and $X_q^{s_2}$ are orthogonal to X_q^μ by the definition of μ (15). Since $H_1H_2 = 0$, $X_q^{s_1}$ and $X_q^{s_2}$ are also perpendicular. Finally, X_q^θ is orthogonal to both $X_q^{s_1}$ and $X_q^{s_2}$ through (26) and (27) and by noting that E_3 is a skew-symmetric matrix. We conclude that the two control distributions $\bar{\Delta}_g$ and $\bar{\Delta}_s$ are orthogonal, so Requirement 3) of *Problem 1* is verified. Moreover, since orthogonal control directions are chosen as the basis for $\bar{\Delta}_g$ and $\bar{\Delta}_s$, each of the formation variables can be individually controlled.

C. Individual Control Laws

The differential map $d\phi_q\colon T_qQ \to T_aA$ is given by

$$d\phi = \frac{1}{N-1}\begin{bmatrix} \frac{N-1}{N}I_2 & \cdots & \frac{N-1}{N}I_2 \\ \frac{1}{s_1-s_2}(q_1 - \mu)^T H_3 & \cdots & \frac{1}{s_1-s_2}(q_N - \mu)^T H_3 \\ (q_1 - \mu)^T H_1 & \cdots & (q_N - \mu)^T H_1 \\ (q_1 - \mu)^T H_2 & \cdots & (q_N - \mu)^T H_2 \end{bmatrix}. \tag{39}$$

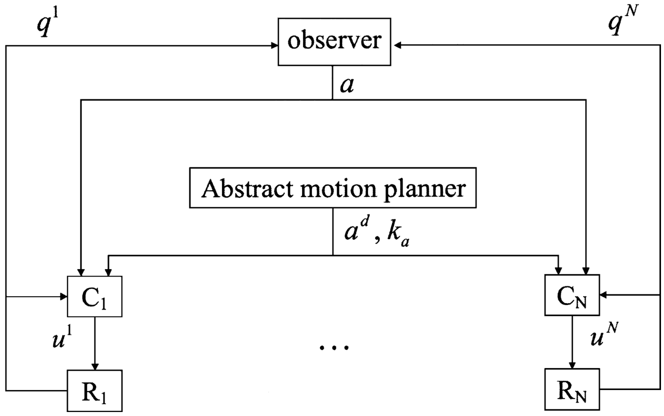

Fig. 1. Control and communication architecture. The control law C_i of each robot R_i is only dependent on its own state q_i and the abstract state a, which is updated by an "observer." An "abstract motion planner" prescribes the desired abstract final state or trajectory a^d and the desired speed of convergence k_a.

The minimum norm vector $\dot{q}$ on TQ which is mapped to a vector field $\dot{a}$ in TA is given by (12). Some simple but rather tedious calculations show that

$$\dot{q} = X_q^{\mu}\dot{\mu} + \frac{s_1 - s_2}{s_1 + s_2}\dot{\theta}X_q^{\theta} + \frac{\dot{s}_1}{4s_1}X_q^{s_1} + \frac{\dot{s}_2}{4s_2}X_q^{s_2}. \tag{40}$$

Note that the controls $\dot{\mu}$, $\dot{\theta}$, $\dot{s_1}$, and $\dot{s_2}$ act on orthogonal directions so one can explicitly control each of the formation variables without affecting the others.

We define the individual controls as projections of the minimum norm vector (40) as follows:

$$u_i = \dot{q}_i = \dot{\mu} + \frac{s_1 - s_2}{s_1 + s_2}H_3(q_i - \mu)\dot{\theta} + \frac{1}{4s_1}H_1(q_i - \mu)\dot{s}_1 + \frac{1}{4s_2}H_2(q_i - \mu)\dot{s}_2. \tag{41}$$

Remark 3: Note that the overall control architecture implementing (41) fits the structure in Fig. 1. Each robot i needs to implement controller C_i, which is only dependent on its own state q_i and the small dimensional abstract state a. An "observer" is responsible for capturing all of the states q_i and calculating the value of the abstract state a at each time instant. This architecture is compatible with our experimental platform, where a blimp equipped with a camera and a processor moves together with the team of ground robots.

Remark 4: $d\phi d\phi^T$ is invertible if and only if $s_1 \neq 0$ and $s_2 \neq 0$, which is also equivalent to the submersion condition in *Definition 2*. Also, the control law (41) is not defined at $s_1 = 0$ and $s_2 = 0$. The abstract behavior on A should be designed so that $s_1 > 0$ and $s_2 > 0$, for all t. A simple inspection of (18) shows that the cases $s_1 = 0$ and $s_2 = 0$ physically correspond to degenerate situations when all of the robots are on the Oy and Ox axis of the formation frame $\{M\}$, respectively.

Remark 5: If $s_1 = s_2$, the derivative of the orientation θ is not defined, as seen from (33). Indeed, in this case, the robots are equally distributed along the axes of the formation frame, and there is no orientation information. When orientation is not important for a certain application, a simpler abstraction might be defined as in Section VI.

Remark 6: Note that if control law (41) is applied to all the robots, then the team undergoes an affine transformation. Indeed, the orbits of the affine group $GA(2)$ in Q are described by $q_i = d + Aq_i^0$, $d \in \mathbb{R}^2$, $A \in GL(2)$, which, by differentiation, gives $\dot{q}_i = \dot{d} + \dot{A}A^{-1}(q_i - d)$, which is the same as (41) with $\mu = d$ and $\dot{A}A^{-1} = (s_1 - s_2)/(s_1 + s_2)H_3\dot{\theta} + 1/(4s_1)H_1\dot{s}_1 + 1/(4s_2)H_2\dot{s}_2$. Any affine transformation is known to preserve collinearity, ratios of distances on lines, and parallelism. Therefore, control law (41) can be used for formations in which preserving properties like the ones mentioned above are important. Even more interesting, it is known that affine transformations preserve the normal distribution. This means that, if the robots are initially normally distributed, by applying the control laws (41), they remain normally distributed. The 5-D abstract state, interpreted as sample mean μ and sample covariance Σ, gives us control over the pose, aspect ratio, and size of the concentration ellipsoid as defined in Section IV-A2.

Remark 7: Another consequence of *Remark 6* is that controllers (41) guarantee collision avoidance under the assumption that the affine map is nonsingular, which is equivalent to $s_1 \neq 0$ and $s_2 \neq 0$.

V. Abstract Behavior

Equation (41) gives the control law which should be implemented by controller C_i, as shown in Fig. 1, if the output function ϕ is defined as in Section IV. At each time instant t, the control system on A acquires all the states q_i, updates its own state a in accordance to (15), (29), (28), (23), and (24), flows along its designed control vector field $\dot{a}$, and disseminates its state a to all the robots.

Assume that the goal is to move the robots from arbitrary initial positions $q_i(0)$ to final rest positions of desired mean μ^d, orientation θ^d, and shape s_1^d and s_2^d.

An obvious choice of the control vector field $\dot{a} = [\dot{\mu}, \dot{\theta}, \dot{s}_1, \dot{s}_2]$ on the abstract manifold A is

$$\begin{aligned} \dot{\mu} &= K_{\mu}(\mu^d - \mu) \\ \dot{\theta} &= k_{\theta}(\theta^d - \theta) \\ \dot{s}_1 &= k_{s_1}\left(s_1^d - s_1\right) \\ \dot{s}_2 &= k_{s_2}\left(s_2^d - s_2\right) \end{aligned} \tag{42}$$

where $K_{\mu} \in \mathbb{R}^{2\times 2}$ is a positive definite matrix and $k_{\theta}, k_{s_{1,2}} > 0$.

More generally, the task might require the robots to follow a desired trajectory $a^d(t) = [\mu^d(t), \theta^d(t), s_1^d(t), s_2^d(t)]$ on A. A control vector field on F can be of the form

$$\begin{aligned} \dot{\mu} &= K_{\mu}\left(\mu^d(t) - \mu(t)\right) + \dot{\mu}^d(t) \\ \dot{\theta} &= k_{\theta}\left(\theta^d(t) - \theta(t)\right) + \dot{\theta}^d(t) \\ \dot{s}_1 &= k_{s_1}\left(s_1^d(t) - s_1(t)\right) + \dot{s_1}^d(t) \\ \dot{s}_2 &= k_{s_2}\left(s_2^d(t) - s_2(t)\right) + \dot{s_2}^d(t). \end{aligned} \tag{43}$$

Note that (42) [or (43)] only guarantees the desired behavior on the abstract manifold A. If the imposed trajectory $a^d(t)$ is bounded at all times, it is easy to see that $a(t)$ is bounded. For the problem to be well defined, we still need to make sure that the internal states are bounded (requirement 5) of *Problem 1*). We have the following.

Proposition 2: If a is bounded, then so are $q_i, i = 1, \dots, N$.

Proof: It is enough to assume boundness of μ, s_1, and s_2 to prove boundness of q_i. Assume that

$$\|\mu - \mu^d\| \le M_\mu \tag{44}$$

$$\left|s_1 - s_1^d\right| \le M_{s_1} \tag{45}$$

$$\left|s_2 - s_2^d\right| \le M_{s_2}. \tag{46}$$

First note that from (28), it follows that:

$$s_1 + s_2 = \frac{1}{N-1}\sum_{i=1}^{N}(q_i - \mu)^T(q_i - \mu) \tag{47}$$

from which, by using (45) and (46), we have

$$\begin{aligned}\|q_i - \mu\| &\le \sqrt{N(s_1 + s_2)}\\ &\le \sqrt{(N-1)\left(M_{s_1} + M_{s_2} + s_1^d + s_2^d\right)}, \quad i = 1, \dots, N.\end{aligned}$$

Finally, using (44), we have

$$\begin{aligned}\|q_i - \mu^d\| &= \|q_i - \mu + \mu - \mu^d\| \le \|q_i - \mu\| + \|\mu - \mu^d\|\\ &\le \sqrt{(N-1)\left(M_{s_1} + M_{s_2} + s_1^d + s_2^d\right)} + M_\mu\end{aligned}$$

which concludes the proof. □

In the stabilization to a point case, the boundness and globally asymptotic convergence to the desired values of the abstract variables are guaranteed by (42). *Proposition 2* proves the boundness of the internal dynamics. We still need to study the equilibria and regions of convergence for each robot. We have the following proposition.

Proposition 3: For any μ^d, θ^d, s_1^d, and s_2^d, the closed-loop system (41), (42), (15), (29), and (28) globally asymptotically converges to the equilibrium manifold $\mu = \mu^d, \theta = \theta^d, s_1 = s_1^d$, and $s_2 = s_2^d$.

Proof: First, from (41), (15), (33), (31), and (32), it is easy to see that the abstract state is in equilibrium ($\dot{a} = 0$) if and only if each robot is in equilibrium ($\dot{q}_i = 0, i = 1, \dots, N$). Therefore, the equilibria of the closed-loop system are sets described by $\mu = \mu^d$, $\theta = \theta^d$, $s_1 = s_1^d$, and $s_2 = s_2^d$.

For the second part, consider the following Lyapunov function defined on Q:

$$V(q) = \frac{1}{2}\|\mu^d - \mu\|^2 + \frac{1}{2}(\theta^d - \theta)^2 + \frac{1}{2}\left(s_1^d - s_1\right)^2 + \frac{1}{2}\left(s_2^d - s_2\right)^2 \tag{48}$$

and consider the derivative of V along the vector field on Q, as follows:

$$\begin{aligned}\dot{V}(q) = &-K_\mu\|\mu^d - \mu\|^2 - k_\theta(\theta^d - \theta)^2\\ &-k_{s_1}\left(s_1^d - s_1\right)^2 - k_{s_2}\left(s_2^d - s_2\right)^2.\end{aligned} \tag{49}$$

Therefore, $\dot{V}(q) \le 0, \forall q \in \mathbb{R}^{2N}$ and $\dot{V} = 0$ if and only if $\mu = \mu^d$, $\theta = \theta^d$, $s_1 = s_1^d$, and $s_2 = s_2^d$, which is also an invariant set for the closed-loop system. According to the global invariant set theorem (LaSalle), to prove the proposition, we only have to prove that $V(q) \to \infty$ as $\|q\| \to \infty$. We prove this by contradiction. Suppose $\|q\| \to \infty$ and there exists some $L > 0$ so that $V(q) < L$. This implies

$$\|\mu - \mu^d\| \le \sqrt{2L}, \quad \left|s_1 - s_1^d\right| \le \sqrt{2L}, \quad \left|s_2 - s_2^d\right| \le \sqrt{2L}.$$

By an argument similar to the one used in the proof of *Proposition 2*, we can conclude that

$$\|q_i - \mu^d\| \le \sqrt{(N-1)\left(s_1^d + s_2^d + 2\sqrt{2L}\right)} + \sqrt{2L}$$

which means that all q_i are bounded, $i = 1, \dots, N$. But $\|q\| \to \infty$ implies that, for at least one $i = 1, \dots, N$, $\|q_i\| \to \infty$. Therefore, we reached a contradiction and the theorem is proved. □

Remark 8: If a robot in the team breaks or simply stops because of the collision with an obstacle, then, in general, the team will not converge to the desired values of the abstract variable, unless the robot stops in a position which is compatible with the desired values of the abstract variable. In an experimental scenario, we propose the following solution. If a robot breaks, it should send a signal to the observer. After receiving the signal, the observer should not take that robot into consideration any more; the individual control laws should be changed by taking $N \leftarrow N - 1$. The convergence of the rest of the team to the desired abstract values is guaranteed by the global asymptotic stability property.

VI. Contractions and Expansions

When orientation is not relevant for a certain application, we can define a simpler three-dimensional (3-D) abstraction as follows. The group g is restricted to the position of the centroid μ. Let $s = s_1 + s_2$ be the new shape variable. Since $H_1 + H_2 = 2I_2$, we have

$$s = \frac{1}{N-1}\sum_{i=1}^{N}(q_i - \mu)^T(q_i - \mu). \tag{50}$$

For the new abstraction (μ, s), it is easy to see that the left invariance property (5) is satisfied. Concerning the bound of the region occupied by the robots, note that (50) implies

$$\|q_i - \mu\| \le \sqrt{(N-1)s} \tag{51}$$

from which we conclude that the N robots described by a 3-D abstract variable $a = (\mu, s)$ are enclosed in a circle centered at μ and with radius $\sqrt{(N-1)s}$. The new control directions become

$$X_q^\mu = \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}, \quad X_q^s = \begin{bmatrix} (q_1 - \mu) \\ \vdots \\ (q_N - \mu) \end{bmatrix}. \tag{52}$$

The decoupling is obvious by the definition of μ. Concerning the internal dynamics, it is easy to check, following the proof of

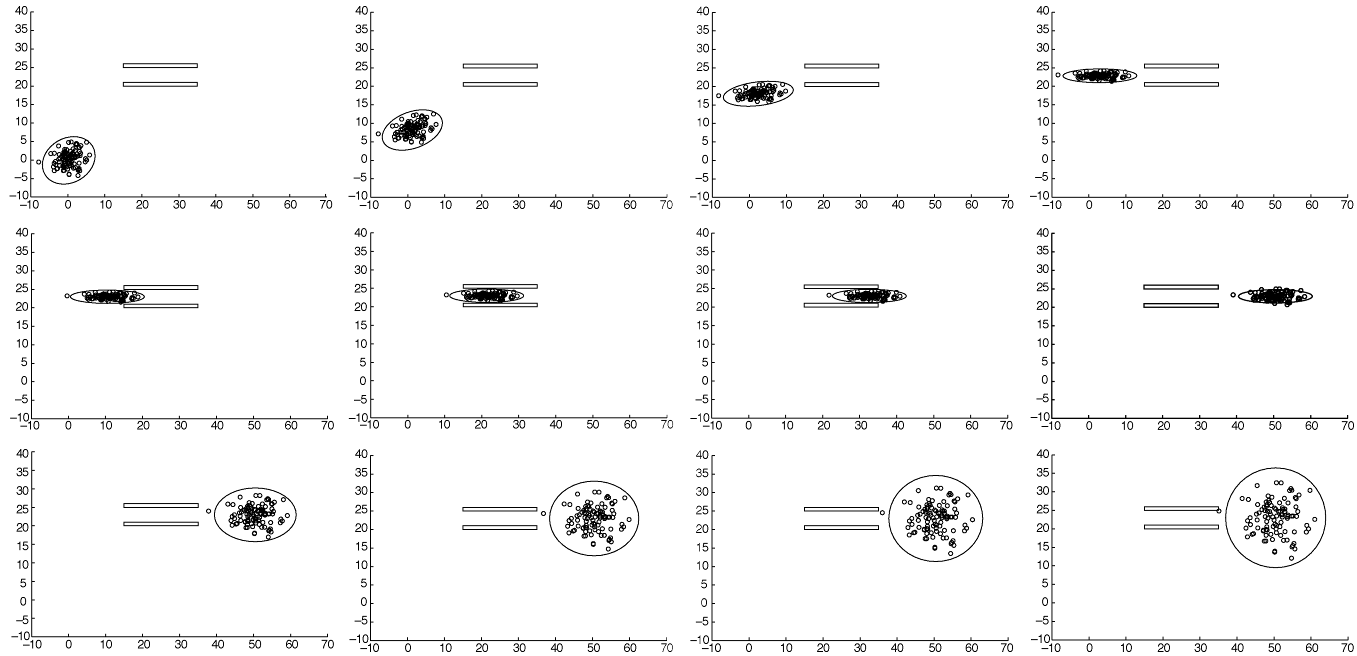

Fig. 2. Ninety-nine of $N = 100$ normally distributed planar robots are driven through a tunnel by designing 5-D controls for the corresponding equiprobability ellipse. First row: the robots gather in front of the tunnel inside an ellipse whose shape and orientation allows passing through the tunnel. Second row: the robots pass through the tunnel; the shape and orientation of the ellipse are kept constant. Third row: the robots spread out by keeping the pose of the ellipse fixed and changing the shape.

Proposition 2, that q_i is bounded if $a = (\mu, s)$ is bounded. The individual control laws (41) become

$$\dot{q}_i = u_i = \dot{\mu} + \frac{q_i - \mu}{2s}\dot{s}, \quad i = 1, \ldots, N. \tag{53}$$

It is easy to prove that control law (53) preserves the orientation of the structure formed by the position vectors q_i in the given inertial frame and scale the pairwise distances by a factor proportional to $\sqrt{s(t)}$. The abstraction is therefore reduced in this case to the position of the centroid and the scale factor of a geometric figure of given shape and orientation determined by the initial positions of the robots.

VII. Simulation Results

This section presents simulations illustrating the theoretical results proved in this paper. First, we show how a team of robots can be driven through a tunnel by designing controls on a 5-D space. For a very large number of initially normally distributed robots, we control a equiprobability ellipsoid. For tens of robots, we control the spanning rectangle. Finally, an expansion example is included.

A. Tunnel Passing

Consider the task of driving a team of robots from arbitrary initial positions through a tunnel of given geometry, and spread out at the end of tunnel. Independent of the number of robots, the problem can be reduced to a 5-D control problem using one of the abstractions proposed in this paper. If the number of robots is of the order of tens, the spanning rectangle as defined in Section IV-A1 can be used. For hundreds and thousands of robots, the spanning rectangle becomes too conservative. In this case, if it is allowed to lose a very small percentage of them and if their initial distribution is assumed normal, we propose the control of a concentration ellipsoid, as described in Section IV-A2.

In both cases, we divide the task into three subtasks.

1) Gather the robots in front of the tunnel in such a shape that they can pass through it.
2) Drive the robots through the tunnel.
3) Spread out at the end of the tunnel.

1) Control Using the Concentration Ellipsoid: Assume $N = 100$ and it is desired that "almost all" of the robots accomplish the task. Assuming that the robots are normally distributed in the initial configuration, they remain normally distributed by applying the control laws (41), according to *Remark 6*. If 99% is an acceptable quantization of "almost all," according to Section IV-A2, the problem can be reduced to a 5-D control problem for a concentration ellipsoid of probability $p = 0.99$.

For the subtask of regrouping in front of the tunnel [step 1)], we use the globally stabilizing controllers (41) and (42). Considering the geometry, position, and orientation of the tunnel, we chose $\mu^d = [3\ 23]$, $\theta^d = 0$, $s_1^d = 10.8574$, and $s_2^d = 0.3518$. The chosen shape corresponds to semiaxes of $\sqrt{cs_1^d} = 10$ and $\sqrt{cs_2^d} = 1.8$ along x and y, respectively. The abstract controller parameters were $K_\mu = 2I_2$, $k_\theta = 2$, and $k_{s_1} = k_{s_2} = 2$. Note that in this first subtask both shape and pose are controlled. Four snapshots from the produced motion are shown in the first row of Fig. 2.

Since the ellipse from 1) is small enough and oriented to fit the tunnel, no shape and orientation control is necessary to accomplish subtask 2). We use trajectory following controllers of type (43) on A to move the ellipse through the tunnel. If we want to uniformly move the ellipse at [50 23] in 1 s while keeping shape and orientation constant, we only have to control μ_x, therefore

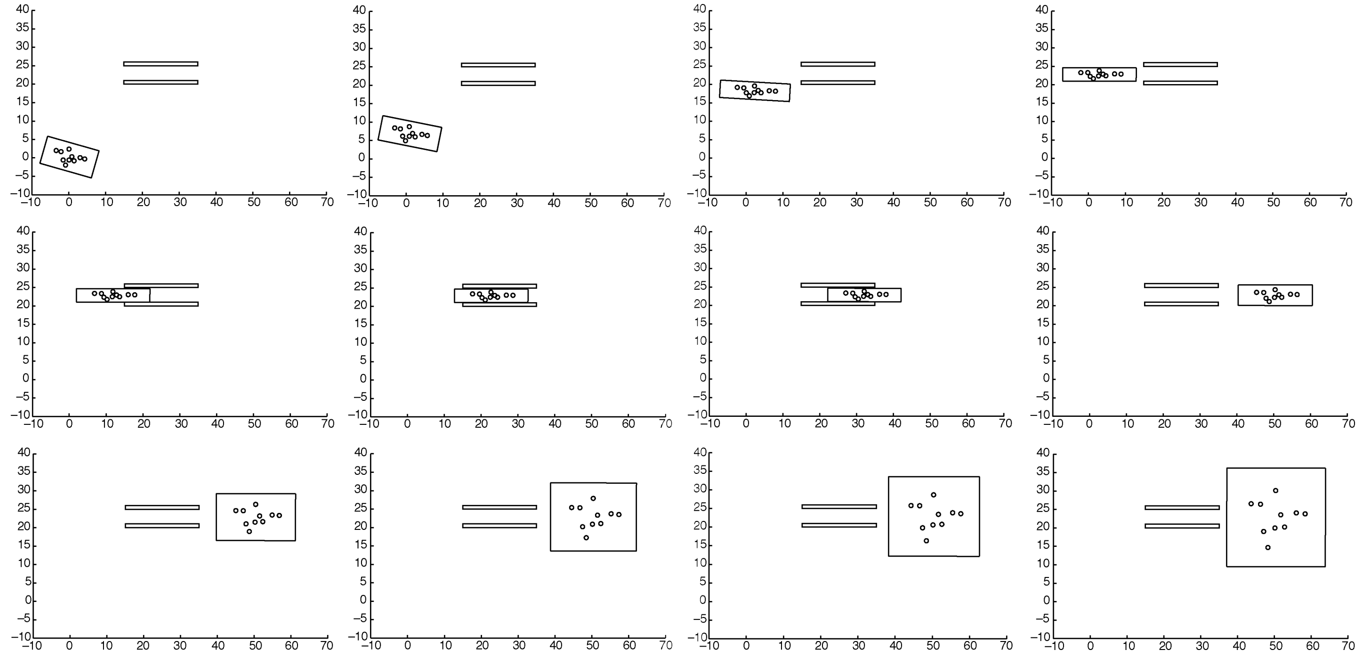

Fig. 3. $N = 10$ planar robots are driven through a tunnel by designing 5-D controls for the corresponding spanning rectangle. First row: the robots gather in front of the tunnel inside a rectangle whose sides and orientation allow passing through the tunnel. Second row: the robots pass through the tunnel; the sides and orientation of the rectangle are kept constant. Third row: the robots spread out by keeping the pose of the rectangle fixed and increasing the lengths of the sides.

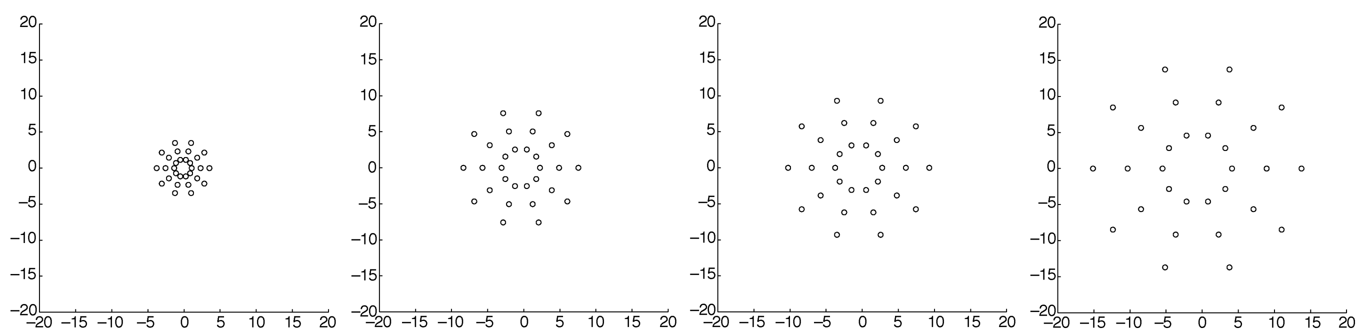

Fig. 4. $N = 30$ robots experiencing an expansion using control law (53). The centroid is kept fixed. Orientation, parallelism, angles, and ratios of lengths are preserved.

$\dot{\mu}_y = \dot{\theta} = \dot{s}_1 = \dot{s}_2 = 0$. We use $\mu_x^d(t) = (1-t)3 + t50$ (therefore $\dot{\mu}_x^d(t) = 47$). The second row of Fig. 2 shows four instants of the generated trajectories. As expected, shape and orientation is preserved, therefore illustrating the control decoupling proved in Section IV-B.

For the third subtask, we illustrate control of shape decoupled from pose, which is maintained constant. We again use the globally stabilizing controllers (41) and (42) with $\dot{\mu} = 0$, $\dot{\theta} = 0$, $s_1^d = s_2^d = 20$, and $k_{s_1} = k_{s_2} = 2$. The obtained expansion is shown in the last row of Fig. 2.

2) Control Using the Spanning Rectangle: If it is now required that all of the robots accomplish the task, we need to use the spanning rectangle as an abstraction. The advantage is that no assumption is being made on the initial distribution of the robots. On the other hand, as stated in Section IV-A3, the spanning rectangle becomes too conservative an estimation of the region occupied by the robots, as the number of robots increases.

We consider $N = 10$. The control procedure follows exactly the one described in Section VII-A1. The control parameters are also the same. The only exception is that, for the first subtask, we used $s_1^d = 11.1111$ and $s_2^d = 0.36$, which correspond to a spanning rectangle of sides $2\sqrt{(N-1)s_1^d} = 20$ and $\sqrt{(N-1)s_2^d} = 3.6$, which is thin enough to fit through the tunnel. The simulation results are shown in Fig. 3.

B. Expansions

Consider $N = 30$ robots, distributed on three concentric circles. We apply the geometric shape preserving control laws (53) to illustrate contractions and expansions. We use global convergence to a point for the abstract state $a = (\mu, s)$. Fig. 4 shows a pure expansion obtained with $\dot{\mu} = 0$ and $\dot{s} = k_s(s^d - s)$ with $k_s = 2$ and $s^d = 400$.

VIII. Conclusion and Future Work

In this paper, we propose a control method for a large number of robots based on an abstraction of the team to a small dimensional manifold invariant to permutations of the robots and whose dimension does not scale with the number of robots. The task to be accomplished by the team suggests a natural feedback control system on the low-dimensional manifold. We focus on planar fully actuated robots and show that it is possible to define an abstraction which has a product structure of a *group* and a *shape*. We also prove that completely decoupled control systems can be designed for group and shape. The individual control laws which are mapped to the desired behavior of the team can be realized by feedback depending only on the robots' current state and the small dimensional state on the abstraction manifold. Future work will be directed toward incorporating more shape variables, include underactuation constraints in the abstraction, extending the results to 3-D environments, and implementing the obtained control architectures in our blimp–car experimental platform.

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