Chapter 13

# Active Filters and Oscillators

Electrical elements have been used to make frequency-selective filters since the early part of the 20th century. These early filters, which utilized only passive inductors, capacitors, and resistors, helped foster the development of the first radio transmitters and receivers by providing circuits with frequency-selective capabilities. The limited range and selectivity of passive $RLC$ circuits were improved somewhat by the invention of the vacuum tube, which permitted the design of filter circuits with feedback. Modern filter design really began with the arrival of high-quality integrated-circuit operational amplifiers in the early 1960s. Modern filters utilize op-amps in combination with $RC$ feedback networks to provide countless filter functions with a wide range of frequency-selective properties.

As demonstrated in Chapter 10, the frequency response of an op-amp feedback circuit can be dramatically changed by the addition of capacitors to its feedback network. This property can be exploited to produce op-amp circuits with well-defined and controllable frequency-response characteristics. Such circuits are part of a family of stable analog feedback circuits called *active filters*. An analog feedback circuit that is intentionally operated outside its stability limit is called an *oscillator*. In this chapter, the characteristics and properties of several active filter and oscillator circuits are examined in detail. The functions performed by these circuits are important in many signal- and information-processing applications. As we shall see, an active op-amp filter can achieve all of its desired properties without the use of inductors. This result is fortunate, because the inductors needed for filter circuits below about 1 MHz tend to be large, difficult to produce in ideal form, and unsuitable for fabrication on an integrated circuit. Filter circuits made solely from op-amps, resistors, and capacitors are readily fabricated in an integrated-circuit environment.

## 13.1 A SIMPLE FIRST-ORDER ACTIVE FILTER

As a prelude to a general discussion of active filters, we first illustrate the basic concepts of active filtering using the circuit of Fig. 13.1(a). This simple filter is a low-pass variety that passes all frequency components below its cutoff frequency and attenuates all frequency components above. (We also recognize this circuit as the modified op-amp integrator of Chapter 2.) Because we are interested in the behavior of the circuit under sinusoidal steady-state, rather than transient, conditions, the circuit is best analyzed in the frequency domain.

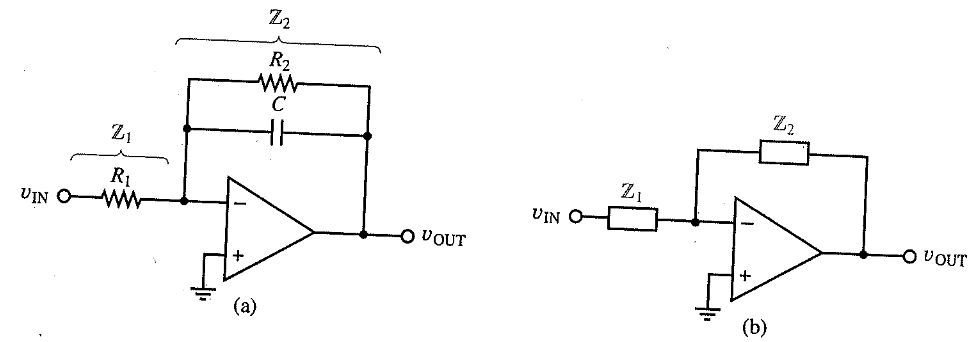


**Figure 13.1** A simple active filter example. (a) Inverting amplifier with feedback "element" $\mathbf{Z}_2 = R_2 \| \mathbf{Z}_C$; (b) equivalent topology of the inverting-amplifier configuration.

The circuit has the same basic topology as the inverting amplifier of Fig. 13.1(b), but in this case, the parallel combination $R_2 \| C$ is used as a feedback element. In the frequency domain, the capacitor behaves as an impedance element of value $\mathbf{Z}_C = 1/j\omega C$. The output of the filter of Fig. 13.1(a) can be found by first expressing $R_2 \| \mathbf{Z}_C$ as a single feedback impedance element of value

$$\mathbf{Z}_2 = R_2 \| \frac{1}{j\omega C} = \frac{R_2}{1 + j\omega R_2 C} \tag{13.1}$$

By analogy to the inverting-amplifier topology of Fig. 13.1(b), the output of the filter becomes

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = -\frac{\mathbf{Z}_2}{\mathbf{Z}_1} = -\frac{R_2}{R_1} \frac{1}{1 + j\omega R_2 C} \tag{13.2}$$

where $\mathbf{Z}_1 = R_1$, and where $v_{OUT}$ and $v_{IN}$ have been represented in sinusoidal phasor form as $\mathbf{V}_{\text{out}}$ and $\mathbf{V}_{\text{in}}$.

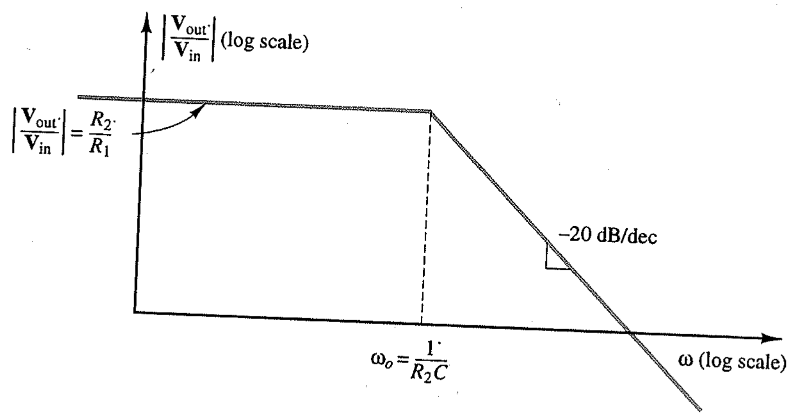


**Figure 13.2** Magnitude Bode plot of the active filter of Fig. 13.1(a). The filter's "cutoff" frequency is designated $\omega_o$.

The transfer function (13.2) has a single pole at $\omega_o = 1/R_2 C$ and a gain of $-R_2/R_1$ well below $\omega_o$. The magnitude Bode plot of this transfer function, shown in Fig. 13.2, has the basic form of a single-pole low-pass filter. As the frequency of the input signal is increased above the "cutoff" frequency $\omega_o$, the filter output decreases at the rate of −20 dB per decade.

The frequency dependency described by the transfer function (13.2) can also be synthesized using the passive $RC$ circuit of Fig. 13.3, for which

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{\mathbf{Z}_C}{R_2 + \mathbf{Z}_C} = \frac{1}{1 + j\omega R_2 C} \tag{13.3}$$

The Bode plot of the latter circuit's response has the same shape as the Bode plot of Fig. 13.2. The advantage of the active filter version of Fig. 13.1(a) over the passive version of Fig. 13.3 is twofold. First, the dc gain of the active filter can be adjusted by changing the ratio $R_2/R_1$. The dc response of the passive circuit of Fig. 13.3 has a fixed value of unity. Second, the output impedance of the active filter of Fig. 13.1(a) is negligibly small; the op-amp functions as a voltage source that drives the output terminal. This feature allows the active filter to drive a load impedance or another stage in a multistage filter cascade without changing the filter characteristics. In contrast, the output impedance of the passive circuit of Fig. 13.3 is equal to $R_2 \| (1/j\omega C)$. This relatively high impedance causes the circuit's output voltage and frequency response to be affected by the characteristics of its load.

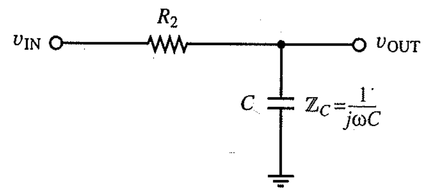


**Figure 13.3** Passive *RC* circuit having the same general frequency response as the active filter of Fig. 13.1(a).

In principle, the transfer function of any passive filter can be synthesized in active form to realize the advantages stated above. Additionally, passive filter circuits that would normally require inductors can be made in active form without the use of inductors. As discussed previously, high-quality inductors are difficult to make in both discrete and integrated environments and are usually avoided in modern active circuit design.

## 13.2 IDEAL FILTER FUNCTIONS

The low-pass filter function described in Section 13.1 is but one of a class of analog filter functions that also includes high-pass, band-pass, and band-reject filters. As its name implies, the high-pass filter passes only those frequency components that lie above some designated cutoff frequency.

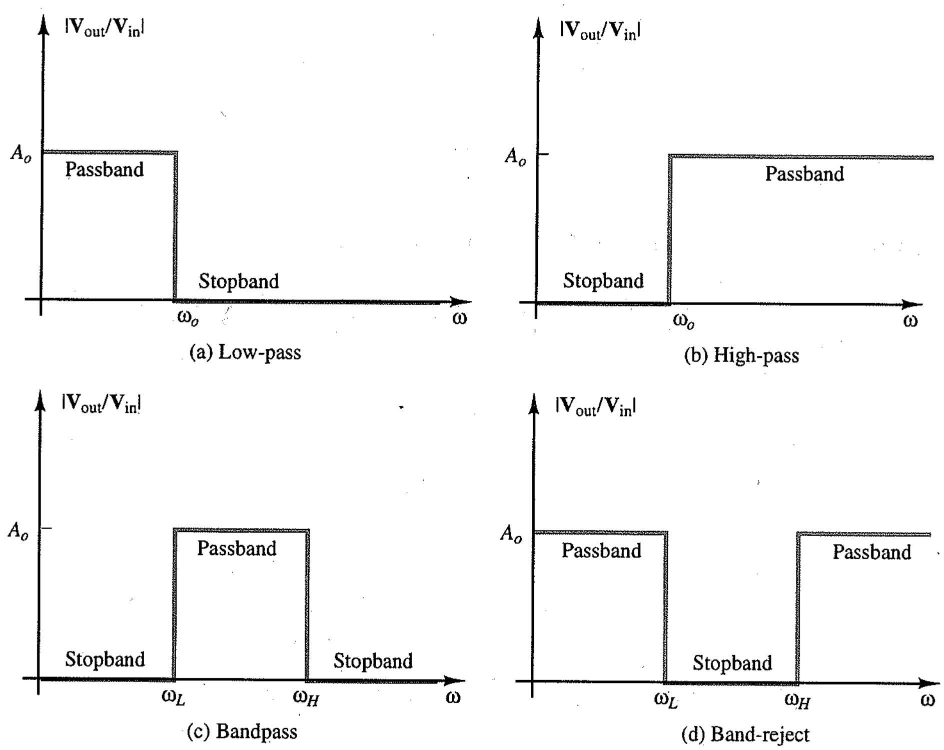


**Figure 13.4** Ideal "brick-wall" responses of (a) low-pass; (b) high-pass; (c) band-pass; and (d) band-reject filters.

The band-pass filter transmits only those frequency components lying within a range specified by upper and lower cutoff limits. The band-reject filter is the inverse of the band-pass filter; it passes only those frequency components lying *outside* some specified frequency range.

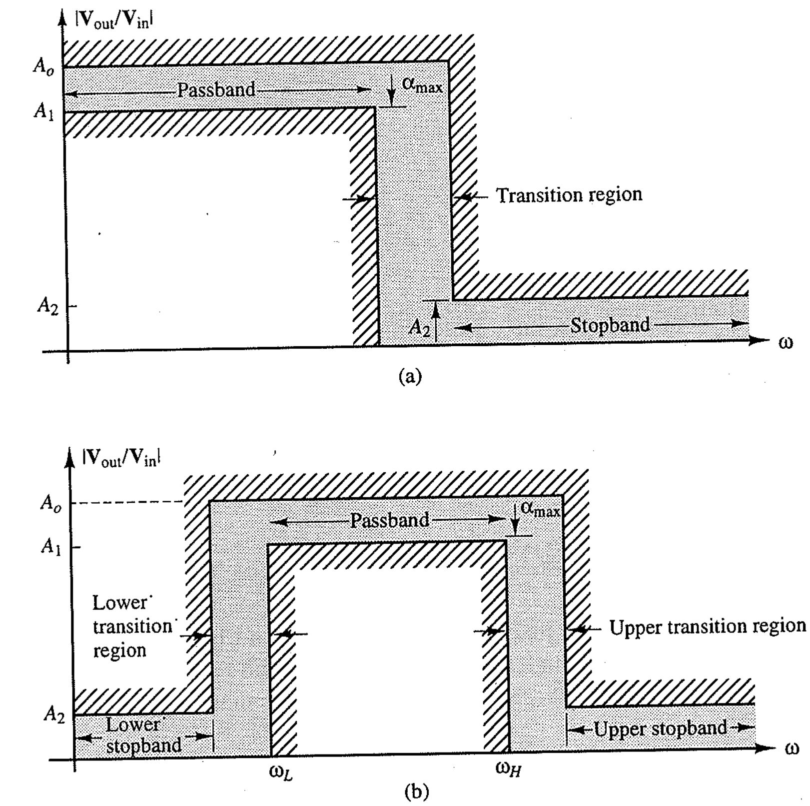


**Figure 13.5** Filter function definitions shown for (a) low-pass filter and (b) band-pass filter.

The basic forms of the transfer function for each of the various filter types are depicted in Fig. 13.4. These perfect, boxlike plots are sometimes called *brick-wall* responses. Each one represents an ideal case in which the filter gain remains constant over frequency regions where signal transmission is desired and falls to zero otherwise. Much of filter design is concerned with approaching these ideal responses as closely as possible while remaining within the practical constraints of part count, cost limitations, and filter complexity. As an example of this concept, consider the simple low-pass op-amp filter of Fig. 13.1(a). Its −20-dB per decade rolloff above $\omega_o$ provides only a very crude approximation to the ideal brick-wall low-pass response of Fig. 13.4(a). The filter is inexpensive and easy to build, however, and is adequate for many applications. A more complex op-amp circuit involving many more components could be constructed to provide a response more closely approaching the ideal, but this choice would result in a larger number of parts, and hence a greater cost per filter.

In order to quantify the degree to which any given filter approaches the ideal "brick-wall" response, it is helpful to define several quantities related to the filter's response curve. These quantities are summarized in Fig. 13.5 using the low-pass and band-pass filters as examples. Similar definitions exist for the high-pass and band-reject filters. The filter's *passband* is defined as the frequency region over which signal transmission is desired. The largest response occurring anywhere within the passband is designated $A_o$. In the ideal case, the filter gain would be equal to $A_o$ throughout the passband. A real filter will always have a gain that changes with frequency, hence the parameter $A_1$ is used to define the lowest value to which the passband gain can fall

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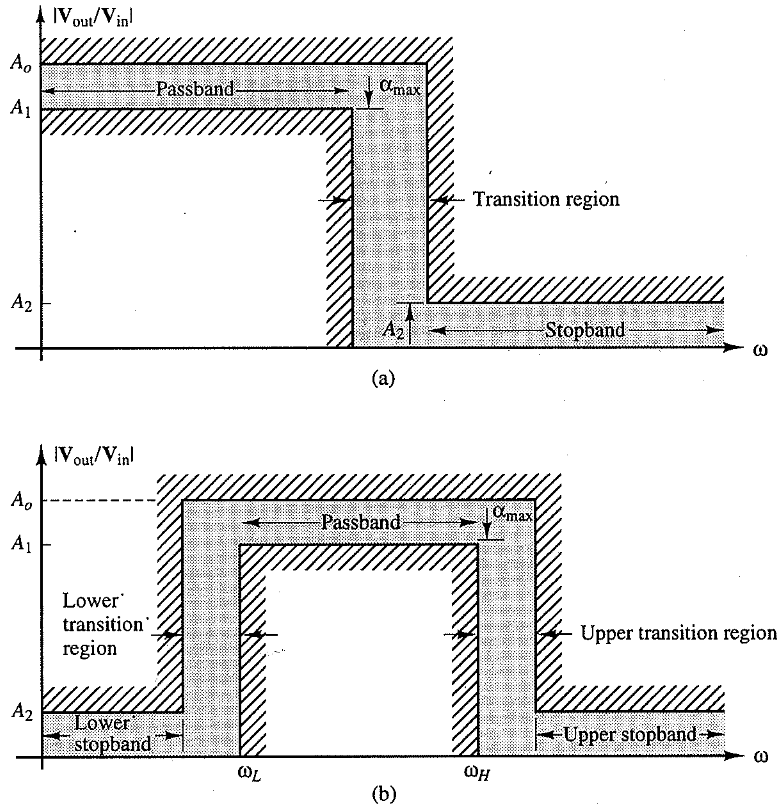


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If an ideal filter could be constructed, its signal transmission would immediately fall to zero outside the passband. In any real filter, some signal transmission always occurs outside the passband. The quantity $A_2$ defines the maximum signal transmission acceptable to the designer outside of the passband. The frequency at which signal transmission first falls to $A_2$ defines the beginning of the filter's *stopband*; the region between the passband and stopband is called the *transition* region. Note that the band-pass filter has two transition regions and two stopbands. Similarly, the band-reject filter has two transition regions and two passbands.

In general, the gain of a filter may lie anywhere between the limits $A_o$ and $A_1$ in the passband; similarly, the gain may lie anywhere below the value $A_2$ within the stopband. The plot of Fig. 13.6(a) shows a low-pass filter response that decreases monotonically from its value of $A_o$ at $\omega = 0$ and reaches the value $A_1$ only once before leaving the passband. The filter response of Fig. 13.6(b) cycles between $A_o$ and $A_1$ several times within the passband and also cycles between zero and $A_2$ within the stopband. The peak passband gain $A_o$ is reached at some frequency other than zero in this second example. Both plots in Fig. 13.6 represent valid low-pass filter responses and reasonable approximations to ideal brick-wall behavior. Each type of response can be produced by an appropriately designed filter circuit.

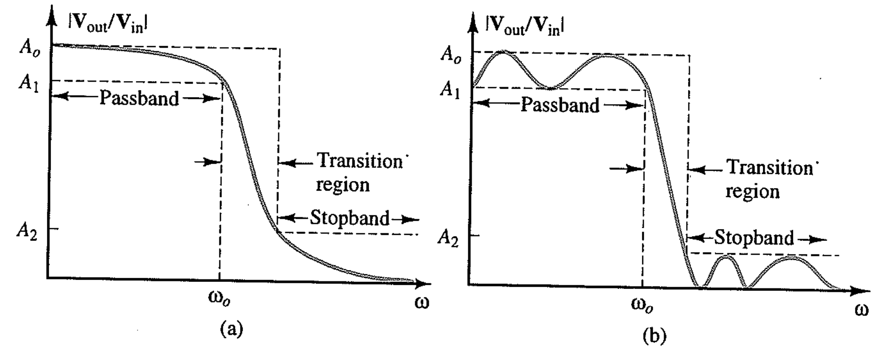



**Figure 13.6** Two possible low-pass filter responses. (a) Gain decreases monotonically as frequency is increased; (b) gain cycles between minimum and maximum limits in the passband and stopband.

## 13.3 SECOND-ORDER FILTER RESPONSES

The low-pass filter of Section 13.1 is an example of a first-order filter. Its single pole in the denominator causes the magnitude $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ to fall off as $1/\omega$, or $-20$ dB/decade, at frequencies well above $\omega_o$. The steep walls of the ideal response of Fig. 13.4(a) are only weakly approximated by the $-20$ dB/decade slope of a first-order filter. A better approximation can be realized by using filters of higher order. The order of a filter is formally defined as the number of poles in the denominator of the transfer function. As a general rule, filters of higher order will have steeper transition region slope(s). The transfer function of a second-order low-pass filter, for example, falls off as $1/\omega^2$, or at $-40$ dB/decade, at frequencies well above its poles. Its slope will be twice as steep as that of a first-order filter, making it a better approximation to the ideal brick-wall response. Transfer functions of even higher order will produce steeper transition-region slopes. In this section, we examine the properties of several second-order filter configurations. In Section 13.4, these filters are used as basic building blocks to synthesize filters of higher order using the technique of *cascading*.

### 13.3.1 The Biquadratic Filter Function

The transfer function of a second-order filter can be described in terms of a ratio two of quadratic polynomials

$$H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = A_o \frac{(1 + j\omega/\omega_1)(1 + j\omega/\omega_2)}{(1 + j\omega/\omega_3)(1 + j\omega/\omega_4)} \tag{13.4}$$

In both the numerator and denominator, the quadratic polynomial has been expressed as the product of two binomials, as in Chapter 9. If the filter is of order 2 or higher, the poles and zeros are generally complex numbers. A transfer function with complex poles and zeros is more readily described using the $s$-plane representation in the sinusoidal steady-state, where $\mathbf{s} = j\omega$. The $s$-plane is defined by a set of real and imaginary axes that are used to plot the real and imaginary components of each pole and zero in the system. In the $s$-plane, the sinusoidal driving frequency of the filter is equivalent to the imaginary-axis variable $\mathbf{s} = j\omega$. If complex numbers $\mathbf{s}_1, \cdots, \mathbf{s}_n$ are used to describe the poles and zeros, the biquadratic transfer function (13.4) takes on the form

$$H(s) = A_o \frac{(1 + \mathbf{s}/\omega_1)(1 + \mathbf{s}/\omega_2)}{(1 + \mathbf{s}/\omega_3)(1 + \mathbf{s}/\omega_4)} \equiv A \frac{(\mathbf{s} + \mathbf{s}_1)(\mathbf{s} + \mathbf{s}_2)}{(\mathbf{s} + \mathbf{s}_3)(\mathbf{s} + \mathbf{s}_4)} \tag{13.5}$$

Equation (13.5) can also be expressed in the general form

$$H(s) = \frac{a_2\mathbf{s}^2 + a_1\mathbf{s} + a_0}{b_2\mathbf{s}^2 + b_1\mathbf{s} + b_0} \tag{13.6}$$

where the coefficients $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ include combinations of the poles $s_1, \cdots, s_4$. This ratio of quadratic polynomials is sometimes called the *biquadratic* transfer function, or simply "biquad." It can be used to describe virtually any second-order filter by appropriate selection of the $a$ and $b$ coefficients. The denominator of the transfer function describing a second-order filter must introduce a factor of $1/\omega^2$ at high frequencies; this criterion can be met by adjusting the coefficients $b_0$, $b_1$, and $b_2$ in (13.6) so that the $s^2$ factor in the denominator dominates at high frequencies. The filter's overall behavior—that is, whether it will be a low-pass, high-pass, band-pass, or band-reject filter—is established by adjusting the numerator coefficients $a_0$, $a_1$, and $a_2$.

### 13.3.2 Second-Order Active Low-Pass Filter

If the coefficients $a_2$ and $a_1$ in Eq. (13.6) are set to zero, the transfer function acquires the form

$$H(s) = \frac{a_0}{b_2\mathbf{s}^2 + b_1\mathbf{s} + b_0} \tag{13.7}$$

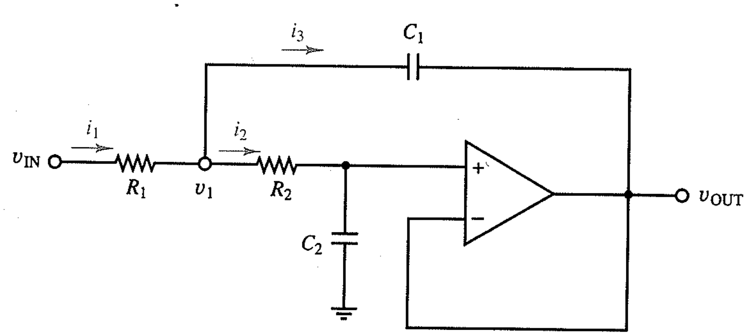



Figure 13.7
Second-order active low-pass filter of the Sallen–Key type.

We recognize this function as that of a second-order low-pass filter. At frequencies near $\mathbf{s} = j\omega \approx 0$, the response approaches the constant value $H(s) = a_0/b_0$. At very high frequencies, the response approaches the limit $H(s) = a_0/b_2\mathbf{s}^2$. Because $\mathbf{s} = j\omega$, this limit falls off as $1/\omega^2$ with a slope of $-40$ dB/decade, as required of a second-order low-pass filter. One filter circuit that has a transfer function of the form (13.7) is shown in Fig. 13.7. The circuit is sometimes called a Sallen–Key filter after its original inventors.[1] Its output as a function of frequency can be found by direct analysis using KVL and KCL. Applying KCL to the $v_1$ node with all currents represented as phasors yields

$$\mathbf{I}_1 = \mathbf{I}_2 + \mathbf{I}_3 \tag{13.8}$$

If the impedance of each capacitor is represented by $\mathbf{Z}_n = 1/j\omega C_n$, Eq. (13.8) can be expressed as

$$\frac{\mathbf{V}_{\text{in}} - \mathbf{V}_1}{R_1} = \frac{\mathbf{V}_1}{R_2 + \mathbf{Z}_2} + \frac{\mathbf{V}_1 - \mathbf{V}_{\text{out}}}{\mathbf{Z}_1} \tag{13.9}$$

The op-amp voltage $v_+$ can be found in terms of $\mathbf{V}_1$ from the complex form of the voltage divider:

$$\mathbf{V}_+ = \mathbf{V}_1 \frac{\mathbf{Z}_2}{R_2 + \mathbf{Z}_2} \tag{13.10}$$

The op-amp output is connected directly to the $v_-$ terminal, thereby forming a voltage follower between $v_+$, $v_-$, and $v_{\text{OUT}}$. This connection forces $v_{\text{OUT}}$ to have the same value as $v_+$, so that Eq. (13.10) becomes

$$\mathbf{V}_{\text{out}} = \mathbf{V}_1 \frac{\mathbf{Z}_2}{R_2 + \mathbf{Z}_2} \tag{13.11}$$

Rearranging Eq. (13.11) results in

$$\mathbf{V}_1 = \mathbf{V}_{\text{out}} \frac{R_2 + \mathbf{Z}_2}{\mathbf{Z}_2} \tag{13.12}$$

Combining Eqs. (13.9) and (13.12) leads to an expression for $\mathbf{V}_{\text{out}}$ as a function of $\mathbf{V}_{\text{in}}$:

$$\frac{\mathbf{V}_{\text{in}}}{R_1} - \mathbf{V}_{\text{out}} \frac{R_2 + \mathbf{Z}_2}{R_1 \mathbf{Z}_2} = \frac{\mathbf{V}_{\text{out}}}{\mathbf{Z}_2} + \mathbf{V}_{\text{out}} \frac{R_2 + \mathbf{Z}_2}{\mathbf{Z}_1 \mathbf{Z}_2} - \frac{\mathbf{V}_{\text{out}}}{\mathbf{Z}_1} \tag{13.13}$$

Equation (13.13) can be solved for $\mathbf{V}_{\text{out}}$, resulting in

$$\mathbf{V}_{\text{out}} \left( \frac{1}{\mathbf{Z}_2} + \frac{R_2 + \mathbf{Z}_2}{\mathbf{Z}_1 \mathbf{Z}_2} - \frac{1}{\mathbf{Z}_1} + \frac{R_2 + \mathbf{Z}_2}{R_1 \mathbf{Z}_2} \right) = \frac{\mathbf{V}_{\text{in}}}{R_1} \tag{13.14}$$

or

$$\mathbf{V}_{\text{out}} = \mathbf{V}_{\text{in}} \frac{\mathbf{Z}_1 \mathbf{Z}_2}{\mathbf{Z}_1 \mathbf{Z}_2 + \mathbf{Z}_1 (R_1 + R_2) + R_1 R_2} \tag{13.15}$$

Substitution of $1/j\omega C_1$ and $1/j\omega C_2$ for $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in Eq. (13.15) results in

$$H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1}{1 - \omega^2 (R_1 R_2 C_1 C_2) + j\omega C_2 (R_1 + R_2)} \tag{13.16}$$

It is possible to factor the denominator of this frequency-dependent transfer function into the standard "product of binomials" form of Chapter 9. For all but a few values of $R_1$, $R_2$, $C_1$,

[1] R. P. Sallen and E. L. Key, "A Practical Method of Designing RC Active Filters," *IRE Transactions on Circuit Theory*, Vol. CT–2, 74–85, March 1955.

and $C_2$, this factoring reveals poles in the denominator that are complex numbers. The transfer function (13.16) can also be represented as $H(s)$ in the $s$-plane if the following substitutions are made:

$$\mathbf{s} = j\omega \tag{13.17}$$

$$\omega_o = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \tag{13.18}$$

and

$$Q = \frac{\omega_o R_1 R_2 C_1 C_2}{C_2(R_1 + R_2)} = \sqrt{\frac{C_1}{C_2}}\left(\frac{\sqrt{R_1 R_2}}{R_1 + R_2}\right) \tag{13.19}$$

Note that the parameter $Q$, called the "quality factor" of the filter, is dimensionless.

By using these substitutions, Eq. (13.16) can be expressed as

$$H(s) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1}{1 + \mathbf{s}^2/\omega_o^2 + \mathbf{s}/\omega_o Q} = \frac{\omega_o^2}{\mathbf{s}^2 + \mathbf{s}(\omega_o/Q) + \omega_o^2} = \frac{\omega_o^2}{(\mathbf{s} - \mathbf{s}_1)(\mathbf{s} - \mathbf{s}_2)} \tag{13.20}$$

On the right-hand side of Eq. (13.20), the denominator has been factored into two complex binomials $(\mathbf{s} - \mathbf{s}_1)$ and $(\mathbf{s} - \mathbf{s}_2)$, where

$$\mathbf{s}_1 = -\frac{\omega_o}{2Q} + \left[\left(\frac{\omega_o}{2Q}\right)^2 - \omega_o^2\right]^{1/2} \tag{13.21}$$

and

$$\mathbf{s}_2 = -\frac{\omega_o}{2Q} - \left[\left(\frac{\omega_o}{2Q}\right)^2 - \omega_o^2\right]^{1/2} \tag{13.22}$$

We recognize Eq. (13.20) as a biquad transfer function in which the $\mathbf{s}^2$ and $\mathbf{s}$ coefficients in the numerator are set to zero. This feature causes the response to be unity at dc ($\mathbf{s} = 0$) and to fall off as $1/\mathbf{s}^2$ at high frequencies. As the expressions (13.21) and (13.22) indicate, $\mathbf{s}_1$ and $\mathbf{s}_2$ are complex conjugates with equal real parts and with imaginary parts of the same magnitude but opposite sign.

The factors $\mathbf{s}_1$ and $\mathbf{s}_2$ represent the poles of the transfer function (13.16). For the case where the $R$ and $C$ values yield a $Q$ less than 0.5, the factor in brackets in Eqs. (13.21) and (13.22) will be positive, so that $\mathbf{s}_1$ and $\mathbf{s}_2$ will be real and Eq. (13.20) can be written in the form

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{\omega_o^2/s_1 s_2}{[(j\omega/\omega_1) + 1][(j\omega/\omega_2) + 1]} = \frac{1}{(1 + j\omega/\omega_1)(1 + j\omega/\omega_2)} \tag{13.23}$$

where $\omega_1 = -s_1$ and $\omega_2 = -s_2$. Equation (13.23) is produced by dividing the numerator and denominator on the right-hand side of Eq. (13.20) by $s_1 s_2$. The right-hand side of Eq. (13.23) is in the standard product of binomials form of Chapter 9, wherein the frequency response is described by two simple, real poles at $\omega_1$ and $\omega_2$. We recognize this transfer function as that of a low-pass filter of second order. At frequencies well below $\omega_1$ and $\omega_2$, its gain is unity. At frequencies well above $\omega_1$ and $\omega_2$, its gain falls off as $1/\omega^2$, that is, at $-40$ dB per decade in frequency.

If $Q = 0.5$ exactly, the factor inside the brackets in Eqs. (13.21) and (13.22) becomes zero. For this case, the poles of the filter coincide at $\omega_o$, reducing the transfer function to

$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{1}{(1 + j\omega/\omega_o)^2} \tag{13.24}$$

If $Q$ is larger than 0.5, the square-root terms in Eqs. (13.21) and (13.22) become imaginary, and the poles become complex-conjugate numbers $\mathbf{s}_1$ and $\mathbf{s}_2$ with real part equal to

$$s_R = -\frac{\omega_o}{2Q} \tag{13.25}$$

and imaginary parts equal to

$$\pm js_I = \pm j[\omega_o^2 - (\omega_o/2Q)^2]^{1/2} \equiv \pm j[\omega_o^2 - s_R^2]^{1/2} \tag{13.26}$$

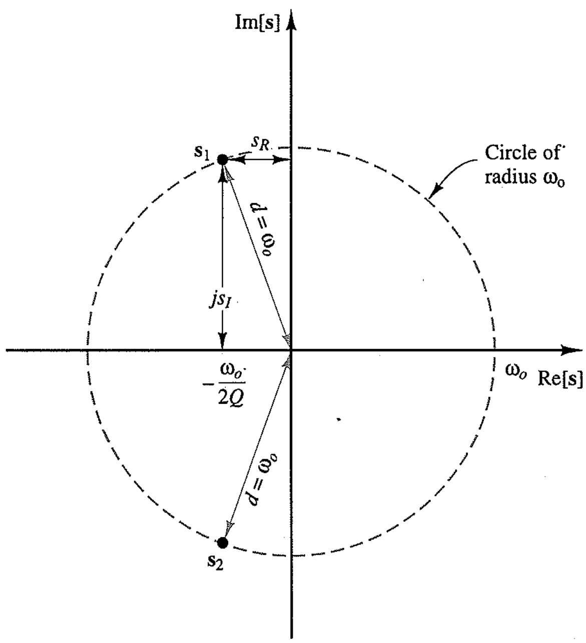
These complex poles $\mathbf{s}_1$ and $\mathbf{s}_2$ are located to the left of the imaginary $s$-axis in the $s$-plane, as shown in Fig. 13.8. Their placement in the left-half plane results because the $s_R$ given by Eq. (13.25) is negative. The radial distance $d$ from the origin to each of the poles $\mathbf{s}_1$ and $\mathbf{s}_2$ in Fig. 13.8 is given by

$$\begin{aligned} d &= |s_R + js_I| \\ &= \{(\omega_o/2Q)^2 + [\omega_o^2 - (\omega_o/2Q)^2]\}^{1/2} = \omega_o \end{aligned} \tag{13.27}$$

As this equation shows, the poles $\mathbf{s}_1$ and $\mathbf{s}_2$, when complex, lie on a circle of radius $\omega_o$ at an angle determined by the value of $Q$. The real and imaginary parts $s_R$ and $s_I$ are not independent. For a given $\omega_o$, specifying $s_R$ automatically specifies $Q$ and $s_I$.

When the poles $\mathbf{s}_1$ and $\mathbf{s}_2$ are imaginary, the magnitude of the transfer function must be expressed as

$$|H(s)| = \left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| = \frac{\omega_o^2}{|\mathbf{s} - \mathbf{s}_1||\mathbf{s} - \mathbf{s}_2|} \tag{13.28}$$




$s_R = -\omega_o/2Q$

$s_I = \pm[\omega_o^2 - (\omega_o/2Q)^2]^{1/2}$

**Figure 13.8** Location of poles $\mathbf{s}_1$ and $\mathbf{s}_2$ for $Q > 0.5$ in the $s$-plane for the low-pass filter transfer function (13.20).

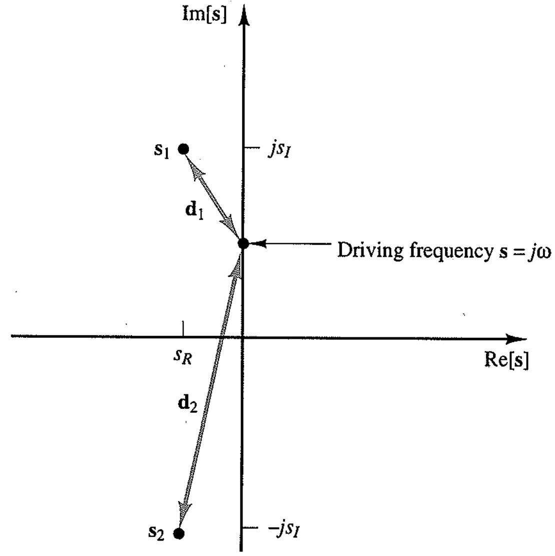



Figure 13.9 Poles of the low-pass filter of Fig. 13.7 in the $s$-plane. The magnitude of $\mathbf{V}_{out}/\mathbf{V}_{in}$ is proportional to the reciprocal of the product $d_1 d_2$.

where $\mathbf{s} = j\omega$. In the $s$-plane representation, the magnitudes $|\mathbf{s} - \mathbf{s}_1|$ and $|\mathbf{s} - \mathbf{s}_2|$ are determined by the distances between the poles $\mathbf{s}_1$ and $\mathbf{s}_2$ and the location $\mathbf{s} = j\omega$ on the imaginary $s$-axis. As depicted in Fig. 13.9, the magnitude $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ becomes

$$|H(s)| = \left|\frac{\mathbf{V}_{out}}{\mathbf{V}_{in}}\right| = \frac{\omega_o^2}{|\mathbf{s} - \mathbf{s}_1||\mathbf{s} - \mathbf{s}_2|} \equiv \frac{\omega_o^2}{d_1 d_2} \tag{13.29}$$

where $d_1$ and $d_2$ are the lengths of vectors $\mathbf{d}_1$ and $\mathbf{d}_2$, respectively.

For small frequencies, such that the driving point $\mathbf{s} = j\omega$ in Fig. 13.8 is located near the origin, the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ have approximately the same length $\omega_o$, and Eq. (13.29) yields $|\mathbf{V}_{out}/\mathbf{V}_{in}| \approx 1$. This situation is depicted in Fig. 13.10(a).

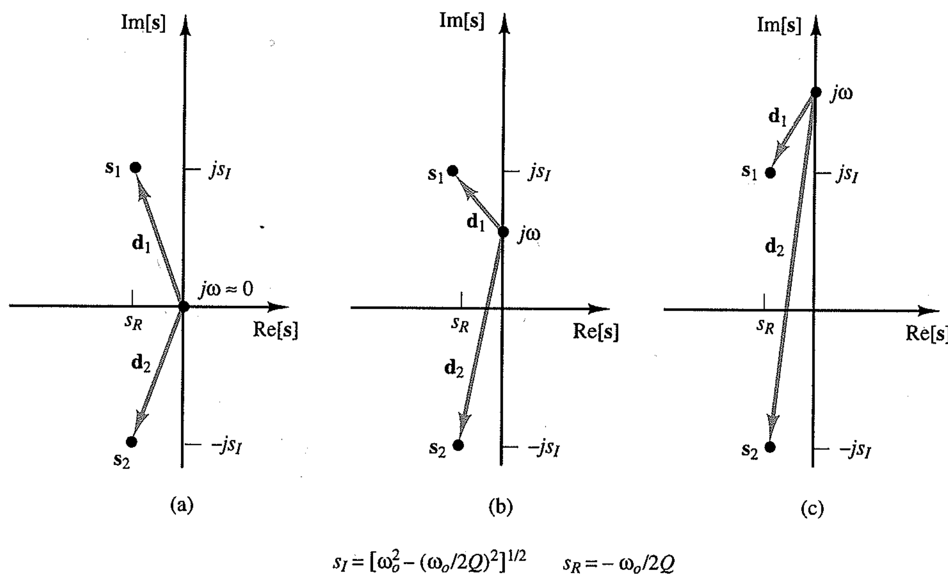



Figure 13.10 Low-pass filter transfer function (13.20) in the $s$-plane. The lengths of vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are shown at three different driving frequencies: (a) $\omega = 0$; (b) $0 < \omega < s_I$; (c) $\omega > s_I$.

As the frequency is raised and $j\omega$ rises up the imaginary $s$-axis, as in Fig. 13.10(b), $\mathbf{d}_2$ becomes longer and $\mathbf{d}_1$ shorter by approximately the same amount, so that $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ remains approximately constant. As $\omega$ exceeds the value $s_I$, both $\mathbf{d}_1$ and $\mathbf{d}_2$ increase in length, and $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ begins to decrease as $1/\omega^2$. The latter case is depicted in Fig. 13.10(c).

We now consider the case of large $Q$. If $Q$ exceeds the value $1/\sqrt{2} = 0.707$, $\mathbf{s}_1$ will lie closer to the imaginary axis than to the real axis, because $s_I$ will be larger than $s_R$. For this case, the product $d_1 d_2$ will be smaller than $\omega_o^2$ as the driving frequency $j\omega$ passes the value $js_I$. The resulting Bode plot of $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ will thus display a rise at the frequency $j\omega = js_I$. At very large values of $Q$ (values of 10 or more), the Bode plot will actually peak sharply as $\mathbf{s} = j\omega$ passes through $js_I$. Note that $js_I$ will be approximately equal to $j\omega_o$ as $Q$ becomes very large.

Figure 13.11 shows the magnitude Bode plot of the second-order low-pass filter function (13.20) for several values of $Q$, including small and large values. In all cases, the roll-off at high frequencies proceeds at $-40$ dB/decade because $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ decreases as $1/\omega^2$. When $Q$ is less than $1/\sqrt{2} = 0.707$, the filter response decreases gradually as the driving frequency $\omega$ passes through $\omega_o$. When $Q$ is greater than 0.707, the filter response peaks above unity as $\omega$ passes through $\omega_o$. When $Q$ is equal to $1/\sqrt{2} = 0.707$, the horizontal portion of the plot extends as far as possible to the right without rising, and the filter's $-3$-dB point lies exactly at $\omega_o$. This condition is sometimes called the *maximally flat* response. The maximally flat transfer function represents a good approximation to the ideal brick-wall response when the filter of Fig. 13.7 is used in stand-alone fashion. When several circuits are cascaded, so as to produce a maximally-flat overall filter response of higher order, the poles of each second-order section are sometimes located so as to produce values of $Q$ other than $1/\sqrt{2}$. This concept is explored in detail in Section 13.4.

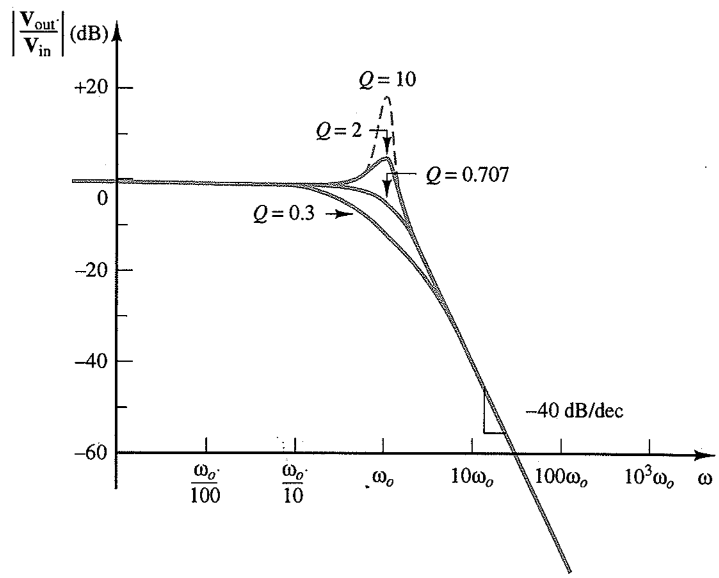



**Figure 13.11** Magnitude plot of the second-order low-pass filter transfer function (13.20) for several values of $Q$. The slope of the Bode plot above $\omega_o$ is equal to $-40$ dB/decade. When $Q = 1/\sqrt{2} = 0.707$, the plot is said to be maximally flat. As $Q$ becomes large, the pole frequency $\mathbf{s}_1$ begins to approach the value $j\omega_o$.

DESIGN

EXAMPLE 13.1

An amplitude-modulated (AM) radio transmission consists of a 530-kHz carrier modulated by an audio signal with frequency components from 300 Hz to 10 kHz. (See Section 4.4.5 for a discussion of amplitude modulation.) The signal is passed through a diode detector, which produces an output consisting of the desired audio signal plus unwanted frequency components at 530 kHz and above. Design a second-order analog filter that will pass the desired audio signal while attenuating the unwanted signals by at least $-60$ dB.

***Discussion.*** The capacitor values determined in this example are not standard values, hence one additional design modification might be to change $C_1$ and $C_2$ to their nearest respective "off-the-shelf" component values, with appropriate changes in $R_1$ and $R_2$. Alternatively, if the capacitors are to be fabricated on an integrated circuit, these nonstandard values can be chosen at fabrication time.

**EXERCISE 13.1** Redesign the filter of Example 13.1 so that $Q = 0.6$. This choice of $Q$ will result in a filter for which the response is not maximally flat. Answer: (one possible design) $R_1 = R_2 = 10\,\text{k}\Omega$; $C_1 = 1.9\,\text{nF}$; $C_2 = 1.3\,\text{nF}$

**13.2** Design a second-order filter with a −3-dB cutoff frequency of 1 kHz and a $Q$ of 1.3. This choice of $Q$ will result in a filter with a peak in its response near $\omega_o$. Answer: (one possible design) $R_1 = R_2 = 5\,\text{k}\Omega$; $C_1 = 83\,\text{nF}$; $C_2 = 12\,\text{nF}$

**13.3** Show that Eq. (13.13) leads to Eq. (13.16).

**13.4** For the pole frequencies defined by Eqs. (13.21) and (13.22), show that $\mathbf{s}_1\mathbf{s}_2 = \omega_o^2$. Express $\mathbf{s}_1$ and $\mathbf{s}_2$ as $s_R \pm js_I$, where $s_R$ and $s_I$ are given by Eqs. (13.25) and (13.26).

**13.5** Plot the magnitude and angle of the low-pass filter transfer function (13.20) as a function of frequency for several values of $Q$. Write a computer program to help with these calculations.

## 13.3.3 Second-Order Active High-Pass Filter

The active circuit of Fig. 13.12 is a second-order high-pass filter of the Sallen–Key type. A high-pass filter transmits frequency components above its cutoff frequency $\omega_o$ and attenuates frequency components below $\omega_o$. The circuit of Fig. 13.12 is the *dual* of the low-pass filter of Fig. 13.7; the locations of all capacitors and resistors are exchanged. Because the basic circuit topology is preserved, the output can be found by exchanging the $R$ and $j\omega C$ terms in the transfer function for the low-pass filter. Performing this operation on Eq. (13.15) yields

$$\mathbf{V}_{\text{out}} = \mathbf{V}_{\text{in}} \frac{R_1 R_2}{R_1 R_2 + R_1(\mathbf{Z}_1 + \mathbf{Z}_2) + \mathbf{Z}_1\mathbf{Z}_2} \tag{13.36}$$

Substitution of $1/j\omega C_1$ for $\mathbf{Z}_1$ and $1/j\omega C_2$ for $\mathbf{Z}_2$ in Eq. (13.36) and some manipulation results in

$$\mathbf{V}_{\text{out}} = \mathbf{V}_{\text{in}} \frac{(j\omega)^2 C_1 C_2 R_1 R_2}{1 - \omega^2(C_1 C_2 R_1 R_2) + j\omega R_1(C_1 + C_2)} \tag{13.37}$$

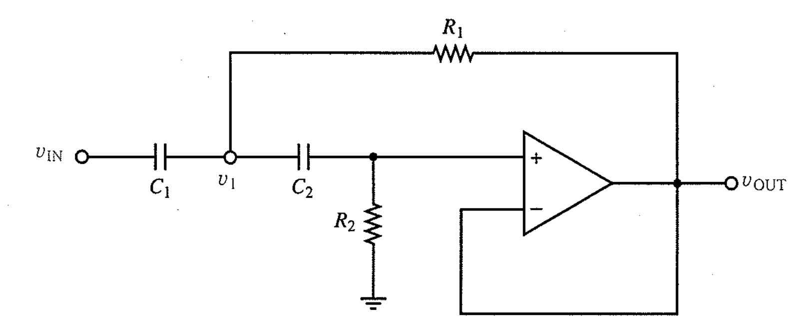



**Figure 13.12** Second-order Sallen–Key active high-pass filter. The slope of the Bode plot below $\omega_o$ is equal to +40 dB/decade.

The transfer function (13.37) can again be represented in the $s$-plane by making the following substitutions:

$$\mathbf{s} = j\omega \tag{13.38}$$

$$\omega_o = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}} \tag{13.39}$$

and

$$Q = \frac{\omega_o R_1 R_2 C_1 C_2}{R_1(C_1 + C_2)} = \sqrt{\frac{R_2}{R_1}}\left(\frac{\sqrt{C_1 C_2}}{C_1 + C_2}\right) \tag{13.40}$$

With these substitutions, Eq. (13.37) becomes

$$H(s) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{\mathbf{s}^2}{\mathbf{s}^2 + \mathbf{s}(\omega_o/Q) + \omega_o^2} = \frac{\mathbf{s}^2}{(\mathbf{s} - \mathbf{s}_1)(\mathbf{s} - \mathbf{s}_2)} \tag{13.41}$$

The roots of the denominator of this expression are again given by Eqs. (13.21) and (13.22), respectively, and Eq. (13.41) again has the form of a biquadratic transfer function. In this case, the numerator consists of a single factor of $\mathbf{s}^2$. At high frequencies, the denominator approaches a limit consisting of a single factor $\mathbf{s}^2$, but this factor is canceled by the factor of $\mathbf{s}^2$ in the numerator. Hence $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ approaches a limit of unity gain at high frequencies. As the driving frequency is reduced well below $\omega_o$, the denominator in (13.41) approaches the constant value $\omega_o^2$. In this case, the factor of $\mathbf{s}^2$ in the numerator causes $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ to fall toward zero at the rate of 40 dB/decade.

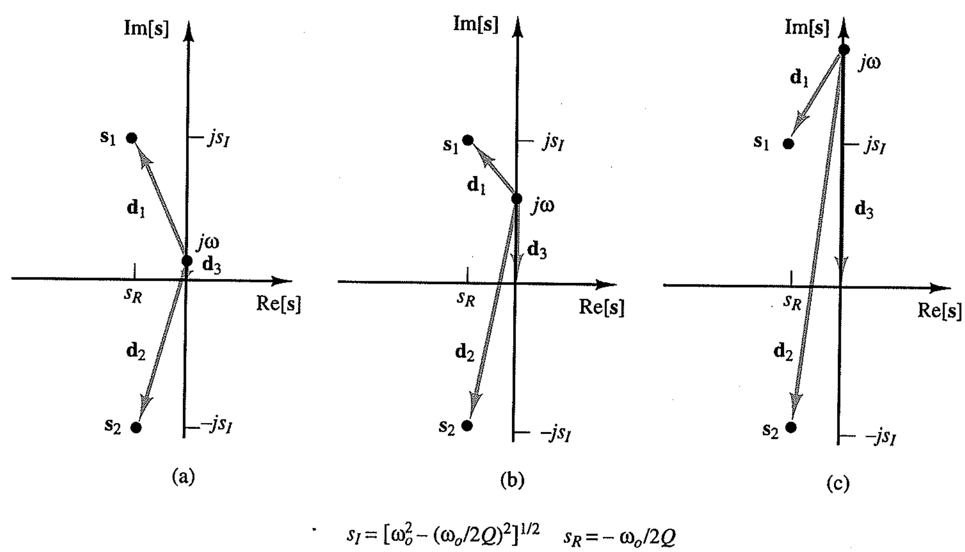



**Figure 13.13** The lengths of vectors $\mathbf{d}_1$, $\mathbf{d}_2$, and $\mathbf{d}_3$ shown at three different frequencies $\omega$ for the second-order high-pass filter function (13.41) with $Q > 0.5$: (a) $\omega \approx 0$; (b) $0 < \omega < s_I$; (c) $\omega > s_I$.

The magnitude of $\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}$ as a function of input frequency can be determined by again examining the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ in the $s$-plane, as in Fig. 13.13. An additional vector $\mathbf{d}_3$, which extends from the location of the driving frequency $j\omega$ to the origin, is needed to represent the factor of $\mathbf{s}^2$ in the numerator of Eq. (13.41). This vector has a length $d_3 = \omega$. When $j\omega$ lies near zero, as in Fig. 13.13(a), vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ have nearly the same length, and $\mathbf{d}_3$ has approximately zero length. The magnitude of the transfer function (13.41) in the limit $j\omega \sim 0$ thus becomes

$$\left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right| = \frac{d_3^2}{d_1 d_2} \approx 0 \tag{13.42}$$

where $d_1$, $d_2$, and $d_3$ represent the lengths of $\mathbf{d}_1$, $\mathbf{d}_2$, and $\mathbf{d}_3$, respectively. If $j\omega$ is increased above zero, $d_2$ will increase by about the same amount that $d_1$ decreases, as in the low-pass filter case. Were it not for $d_3$, this relationship would again keep $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ constant for $\omega < s_I$. The length $d_3$ increases with $j\omega$, however, so that $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ for the high-pass filter increases as the square of $\omega$. As $\omega$ approaches $s_I$, $d_1$ reaches a minimum. Well above $\omega = s_I$, the lengths $d_1$, $d_2$, and $d_3$ all approach the same value, so that the magnitude of $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ approaches unity. Note that if $s_R \ll s_I$, $s_I$ will be approximately equal to $\omega_o$, as seen from Eq. (13.26).

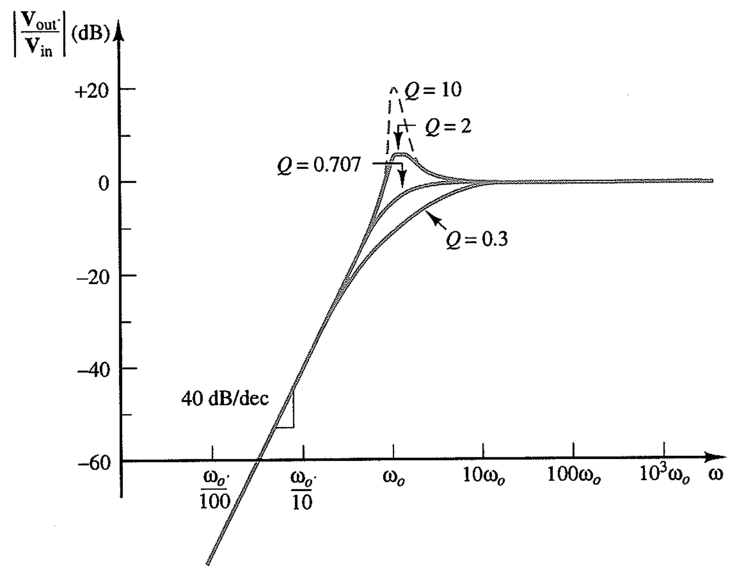



Figure 13.14 Magnitude plot of the second-order high-pass filter transfer function (13.41) for several values of $Q$. When $Q = 1/\sqrt{2} = 0.707$, the plot is said to be maximally flat.

The magnitude Bode plots of this filter for several values of $Q$ are shown in Fig. 13.14. Each plot has a low-frequency +40 dB/decade slope (the response increases as $\omega^2$), a corner frequency near $\omega_o$, and a flat region above $\omega_o$. As in the low-pass filter case, if $Q < 0.5$, the poles $\mathbf{s}_1$ and $\mathbf{s}_2$ become real and the filter function (13.41) can be expressed by

$$H(j\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{(j\omega)^2/\omega_1\omega_2}{(1 + j\omega/\omega_1)(1 + j\omega/\omega_2)} \tag{13.43}$$

where $\omega_1 = \mathbf{s}_1$, $\omega_2 = \mathbf{s}_2$, and $\omega_1\omega_2 = \omega_o^2$. Equation (13.43) is in the product-of-binomials form introduced in Chapter 9. For the case $Q = 0.5$, the poles $\omega_1$ and $\omega_2$ coincide at $\omega_o$.

We note in Fig. 13.14 that when $Q$ is small, the magnitude of the response increases gradually as $\omega$ pass through $\omega_o$. When $Q$ is equal to $1/\sqrt{2} = 0.707$, the response becomes maximally flat. For larger values of $Q$, a peak appears in the response at $\omega = \omega_o$.

**EXERCISE**

**13.6** Design a second-order high-pass filter with parameters $\omega_o = 1$ kHz and $Q = 0.707$.

**13.7** Draw the angle plot $\measuredangle\, \mathbf{V}_{out}/\mathbf{V}_{in}$ as a function of frequency for the high-pass filter transfer function (13.41) for several values of $Q$.

**13.8** Show that Eq. (13.36) leads to Eq. (13.37).

### 13.3.4 Second-Order Active Band-Pass Filter

The op-amp circuit shown in Fig. 13.15 is a *second-order active band-pass* filter. It transmits only those frequency components contained within its passband and attenuates both low-frequency and high-frequency components that lie outside this range. The band-pass behavior of the circuit can be confirmed qualitatively by examining the circuit in the limits of zero and infinite frequency. In the limit $j\omega \to 0$, capacitors $C_1$ and $C_2$ behave as open circuits, and $v_{IN}$ is effectively disconnected from the op-amp terminals. The remaining dc portion of the circuit has the form of a follower with zero input. In the limit $j\omega \to \infty$, capacitors $C_1$ and $C_2$ behave as short circuits, and the circuit functions as an inverting amplifier with a feedback impedance of zero. Because the gain of an op-amp inverter is proportional to its feedback impedance, the resulting output equals zero regardless of input.

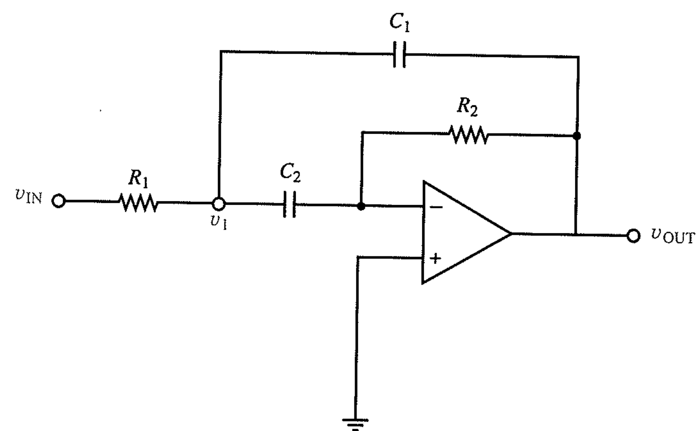



**Figure 13.15** Second-order active bandpass filter of the Sallen–Key type.

A mathematical expression for the transfer function of this circuit can be derived from KVL and KCL. The op-amp, $C_2$, and $R_2$ form an inverting amplifier between $v_{OUT}$ and $v_1$. The voltage $v_{OUT}$ can thus be expressed in the frequency domain by

$$\mathbf{V}_{out} = -\frac{R_2}{\mathbf{Z}_2}\mathbf{V}_1 \tag{13.44}$$

where $\mathbf{Z}_2 = 1/j\omega C_2$. Both $\mathbf{V}_{in}$ and $\mathbf{V}_{out}$ contribute to $\mathbf{V}_1$; hence $\mathbf{V}_1$ can be found using superposition and the complex form of the voltage divider. Alternately setting $\mathbf{V}_{out}$ and $\mathbf{V}_{in}$ to zero with $v_-$ assumed to be at virtual ground potential yields a value for $\mathbf{V}_1$:

$$\mathbf{V}_1 = \mathbf{V}_{in}\frac{\mathbf{Z}_1\|\mathbf{Z}_2}{R_1 + \mathbf{Z}_1\|\mathbf{Z}_2} + \mathbf{V}_{out}\frac{R_1\|\mathbf{Z}_2}{\mathbf{Z}_1 + R_1\|\mathbf{Z}_2} \tag{13.45}$$

Substituting Eq. (13.45) for $\mathbf{V}_1$ into Eq. (13.44) results in

$$\mathbf{V}_{out} = -\frac{R_2}{\mathbf{Z}_2}\left(\mathbf{V}_{in}\frac{\mathbf{Z}_1\|\mathbf{Z}_2}{R_1 + \mathbf{Z}_1\|\mathbf{Z}_2} + \mathbf{V}_{out}\frac{R_1\|\mathbf{Z}_2}{\mathbf{Z}_1 + R_1\|\mathbf{Z}_2}\right) \tag{13.46}$$

Equation (13.46) can be simplified by substituting appropriate expressions for each parallel combination of impedances and by moving the factor of $R_2/\mathbf{Z}_2$ inside the parentheses:

$$\begin{aligned}\mathbf{V}_{out} &= -\left[\mathbf{V}_{in}\frac{\mathbf{Z}_1 R_2/(\mathbf{Z}_1 + \mathbf{Z}_2)}{R_1 + \mathbf{Z}_1\mathbf{Z}_2/(\mathbf{Z}_1 + \mathbf{Z}_2)} + \mathbf{V}_{out}\frac{R_1 R_2/(R_1 + \mathbf{Z}_2)}{\mathbf{Z}_1 + R_1\mathbf{Z}_2/(R_1 + \mathbf{Z}_2)}\right] \\ &= -\left(\frac{\mathbf{V}_{in}\mathbf{Z}_1 R_2 + \mathbf{V}_{out}R_1 R_2}{R_1\mathbf{Z}_1 + R_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_2}\right)\end{aligned} \tag{13.47}$$

Solving Eq. (13.47) for $\mathbf{V}_{\text{out}}$ results, after some algebra, in

$$\mathbf{V}_{\text{out}} = \frac{-\mathbf{Z}_1 R_2}{R_1\mathbf{Z}_1 + R_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_2 + R_1R_2}\mathbf{V}_{\text{in}} \tag{13.48}$$

Substituting $1/j\omega C_1$ for $\mathbf{Z}_1$ and $1/j\omega C_2$ for $\mathbf{Z}_2$ into Eq. (13.48) produces the desired transfer function:

$$\begin{aligned} H(j\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} &= \frac{-R_2/j\omega C_1}{R_1/j\omega C_1 + R_1/j\omega C_2 + (1/j\omega C_1)(1/j\omega C_2) + R_1R_2} \\ &= \frac{-R_2(j\omega C_2)}{j\omega R_1(C_1+C_2) + 1 + (j\omega C_1)(j\omega C_2)R_1R_2} \end{aligned} \tag{13.49}$$

Dividing the numerator and denominator of Eq. (13.49) by $R_1R_2C_1C_2$ and expressing $j\omega$ as $\mathbf{s}$ results in

$$H(s) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{-j\omega R_2C_2/R_1R_2C_1C_2}{\frac{(j\omega C_1)(j\omega C_2)R_1R_2 + j\omega R_1(C_1+C_2) + 1}{R_1R_2C_1C_2}} = \frac{-\omega_o^2 R_2C_2\mathbf{s}}{\mathbf{s}^2 + \mathbf{s}(\omega_o/Q) + \omega_o^2} \tag{13.50}$$

where
$$\omega_o = \frac{1}{\sqrt{R_1R_2C_1C_2}} \tag{13.51}$$

and
$$Q = \frac{\omega_o R_1R_2C_1C_2}{R_1(C_1+C_2)} = \sqrt{\frac{R_2}{R_1}}\left(\frac{\sqrt{C_1C_2}}{C_1+C_2}\right) \tag{13.52}$$

Equation (13.50) has the form of a biquad transfer function in which the numerator contains a single factor of $\mathbf{s}$. As the frequency is reduced well below $\omega_o$, the denominator approaches a constant value of $\omega_o^2$. The factor of $\mathbf{s}$ in the numerator thus causes $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ to fall toward zero at the rate of 20 dB/decade as the frequency is reduced. As the frequency is increased well above $\omega_o$, the denominator of $H(s)$ approaches the limit $\mathbf{s}^2$. The factor of $\mathbf{s}$ in the numerator cancels one factor of $\mathbf{s}$ in the denominator, leaving a single factor of $\mathbf{s}$ in the denominator that causes $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ to be reduced as $1/s$ (i.e., at the rate of $-20$ dB/decade) as the frequency is increased. At the passband center frequency $\omega_o$, the magnitude $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ can be found by substituting $\omega = \omega_o$ into Eq. (13.49):

$$\begin{aligned} \left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right|_{\omega=\omega_o} &= \left|\frac{-j\omega_o R_2C_2}{j\omega_o R_1(C_1+C_2) + 1 - \omega_o^2 R_1R_2C_1C_2}\right| \\ &= \frac{R_2C_2}{R_1(C_1+C_2)} \ . \end{aligned} \tag{13.53}$$

where $\omega_o = 1/\sqrt{R_1R_2C_1C_2}$. Given the expression (13.52) for $Q$, Eq. (13.53) becomes

$$\left|\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}}\right|_{\omega=\omega_o} = \sqrt{\frac{R_2C_2}{R_1C_1}}\,Q \tag{13.54}$$

As Eq. (13.54) suggests, $|\mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}|$ at $\omega = \omega_o$ is dependent on $Q$. Analysis of Eq. (13.50) in the $s$-plane yields the magnitude plots of Fig. 13.16, shown for several values of $Q$ and values of $R$ and $C$ such that $R_2C_2/R_1C_1 = 100$.

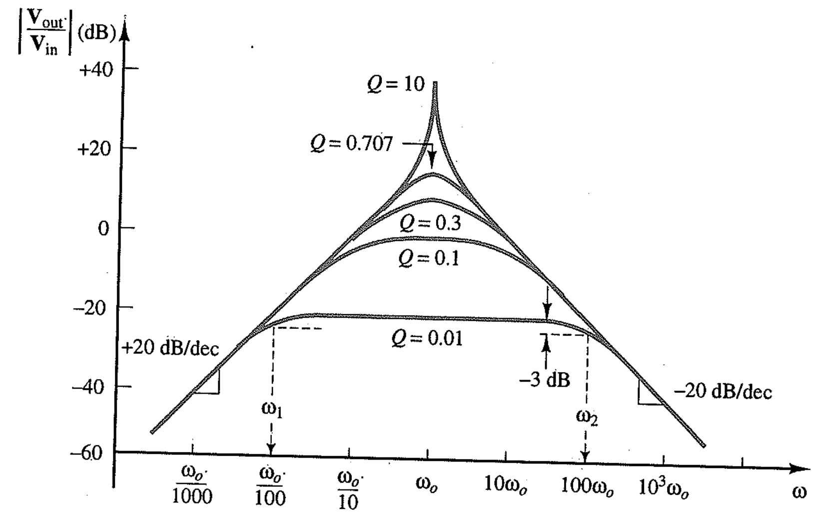



**Figure 13.16** Magnitude plot of the band-pass filter transfer function (13.50) for several values of $Q$. The components $R_1$, $C_1$, $R_2$, and $C_2$ have been chosen so that $R_2C_2/R_1C_1 = 100$. The locations of the passband limits $\omega_1$ and $\omega_2$ are shown for the case $Q = 0.01$.

For any value of $Q$, the two frequencies $\omega_1$ and $\omega_2$ at which the output falls by a factor of $1/\sqrt{2} \equiv -3$ dB from its value at $\omega = \omega_o$ are called the *half-power* frequencies of the band-pass filter. The difference $\omega_2 - \omega_1$ is called the *bandwidth* of the filter. The relationship between $\omega_o$, $\omega_1$, $\omega_2$, and the bandwidth of the filter for the specific case $Q = 0.01$ is illustrated in Fig. 13.16. For large $Q$, analysis of the transfer function in the $s$-plane shows the half-power frequencies to be located at $\omega_o \pm \omega_o/2Q$ on either side of $\omega_o$, as we now show.

The denominator of Eq. (13.50) can be factored into two binomials $(\mathbf{s} - \mathbf{s}_1)$ and $(\mathbf{s} - \mathbf{s}_2)$ and the magnitudes of these binomials represented by the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ shown in Fig. 13.17. The vector $\mathbf{d}_3$ in Fig. 13.17 represents the factor of $\mathbf{s}$ in the numerator of Eq. (13.50). As shown in Fig. 13.17(a), the vector length $d_1$ reaches its minimum value at the frequency $\omega = s_I$. If $s_R$ is small compared to $s_I$ (i.e., large $Q$), then

$$s_I = \left[\omega_o^2 - \left(\frac{\omega_o}{2Q}\right)^2\right]^{1/2} \approx \omega_o \tag{13.55}$$

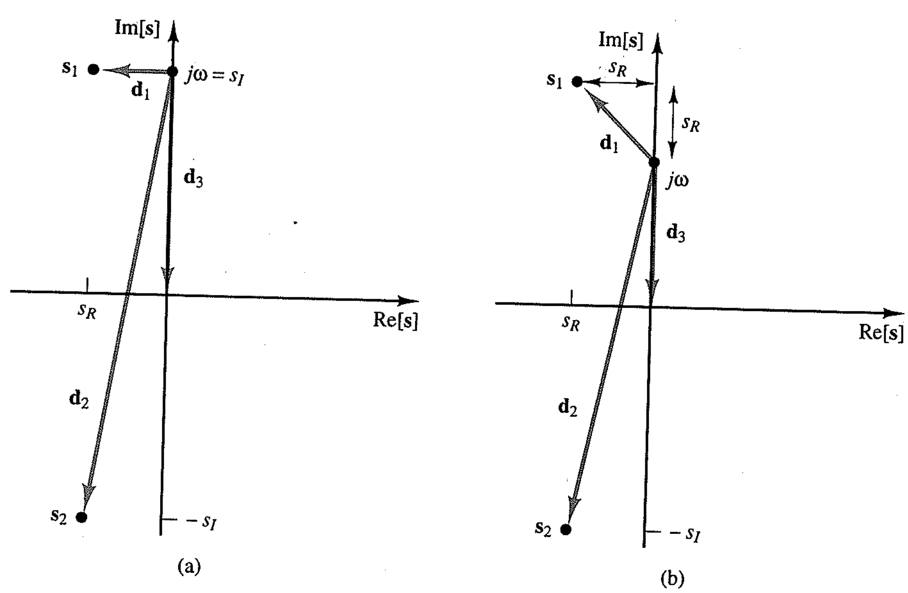



**Figure 13.17** The $s$-plane representation of the poles of the second-order band-pass filter of Fig. 13.15: (a) $\omega = s_I$; (b) $\omega = s_I - |s_R|$.

At this frequency, the magnitude of $\mathbf{V}_{out}/\mathbf{V}_{in}$, which is proportional to $d_3/d_1d_2$, reaches its maximum value. If $\omega$ is decreased to the value $(s_I - s_R)$, as in Fig. 13.17(b), $d_1$ will increase by a factor of $\sqrt{2}$. If $s_R$ is small compared to $s_I$, the vector lengths $d_2$ and $d_3$ will be decreased only slightly at this new frequency. As a result of these combined effects, $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ will fall by a factor of $1/\sqrt{2}$ from its value at $\omega = s_I \approx \omega_o$. The half-power frequency $\omega_1$, therefore, is equal to

$$\omega_1 = s_I - s_R \tag{13.56}$$

Note from the denominator of Eq. (13.50) that $|s_R|$ is equal to $\omega_o/2Q$.

Given the result (13.55), the half-power frequency becomes

$$\omega_1 \approx \omega_o - \frac{\omega_o}{2Q} \tag{13.57}$$

This same reasoning can be applied to the case $\omega_2 = (s_I + s_R)$ to yield

$$\omega_2 \approx \omega_o + \frac{\omega_o}{2Q} \tag{13.58}$$

The resulting bandwidth of the filter for the case $s_R \ll s_I$ is thus equal to

$$\mathrm{BW} = \omega_2 - \omega_1 = \frac{\omega_o}{Q} \tag{13.59}$$

Note that a small $s_R$ is equivalent to a large $Q$, since $Q = \omega_o/2s_R$.

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**EXERCISE**

**13.9** Derive the result (13.54) by substituting $s = j\omega_o$ into Eq. (13.50).

**13.10** Synthesize a second-order band-pass filter with a $Q$ of 100 and a center frequency of 10 kHz. What is the magnitude of the response at $\omega = \omega_o$? What is the bandwidth of the filter?

**13.11** Design a band-pass filter using the specifications of Exercise 13.10 so that the gain is equal to +60 dB at $\omega_o$. What is the bandwidth of your new design?

**13.12** By analyzing the transfer function (13.50) in the $s$-plane, show that the band-pass filter of Fig. 13.15 produces responses with the magnitude plots of Fig. 13.16.

**13.13** Draw the angle plot $\measuredangle\, \mathbf{V}_{out}/\mathbf{V}_{in}$ as a function of frequency for the band-pass filter transfer function (13.49) for several values of $Q$. Write a computer program to help with these calculations.

**13.14** Show that Eq. (13.48) follows from Eq. (13.47).

**13.15** Arrive at the result (13.48) using KCL at the $v_1$ node, rather than the superimposed complex voltage-divider expression (13.45).

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## 13.4 ACTIVE FILTER CASCADING

The usefulness of active filters becomes apparent when two or more are cascaded together to produce filter transfer functions of increased order or complexity. As mentioned in Section 13.1, the output of any one active op-amp filter stage will appear as a voltage source to the input of the next stage, so that interstage loading problems are virtually nonexistent. The overall transfer function of an active filter cascade thus will be equal to the simple product of the transfer functions of each of its individual stages. A filter of order higher than 2 is easily synthesized by simply cascading several one- or two-pole filters in series. The poles of each component filter are appropriately chosen such that the desired overall response is achieved.

In general, the higher the order of a filter cascade, the more closely its transfer function can be made to approach one of the ideal "brick-wall" responses of Fig. 13.4. In this section, we examine the techniques required to accomplish this task. Although we shall focus primarily on the low-pass filter, the concepts introduced apply equally well to high-pass, band-pass, and band-reject filters.